Analysis of One Dimensional Non Convex Variational Problems with Restrictions On the Derivative

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August 22, 2006

Abstract

In this work we propose a particular method for analyzing non convex variational problems given in in the general form:

$$\min_{u} \int_0^1 f\left(x, y\left(x\right), y'\left(x\right)\right)\, dx$$

where the derivative of the admissible functions y is constrained by the following ineqaulities:

$$\alpha\left(x, y\left(x\right)\right) \leq y'\left(x\right) \leq \beta\left(x, y\left(x\right)\right) \text{ for every } x \in [0,1]\,.$$

The method proposed here allows us to study general cases where the integrand f admits a polynomial description in y' without any assumption about constraint functions: α and β. This method determines the existence of minimizers, even in the cases where the integrand f lacks of convexity. The method is based on the general theory of Young measures and classical results on algebraic moments theory.

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where the derivative of the admissible functions y is constrained by the following ineqaulities:

$$\alpha\left(x, y\left(x\right)\right) \leq y'\left(x\right) \leq \beta\left(x, y\left(x\right)\right) \text{ for every } x \in [0,1]\,.$$

The method proposed here allows us to determine the existence of minimizers whenever the integrand f admits a polynomial description in y'. It is remarkable that this method does not need any assumption about the constraint functions: α and β and it also works well in those cases in which the integrand f does not have a convex dependence in y'. The analysis presented here is based on the general theory of Young measures and classical results on algebraic moments theory.

1 Introduction

In this paper we will analyze the general family of one dimensional variational problems given in the following form:

$$\begin{array}{l} \min_y I\left(y\left(x\right)\right)=\int_a^b f\left(x,y\left(x\right),y'\left(x\right)\right)\,dx \\ s.t. \\ \alpha\left(x,y\left(x\right)\right)\leq y'\left(x\right)\leq\beta\left(x,y\left(x\right)\right)\quad for\ every\quad x\in\left[a,b\right] \\ and\quad Boundary\quad Conditions: y\left(a\right)=y_a,\quad y\left(b\right)=y_b \end{array} \tag{1}$$

where the admissible functions y are supposed to belong to a proper Sobolev space according to the particular integrand f. We will study the variational Problem 7 when the integrand f is given by the general expression

$$f\left(x,y,\lambda\right)=\sum_{k=0}^{n}c_k\left(x,y\right)\lambda^k \tag{2}$$

which means that f is an arbitrary algebraic polynomial in the derivative y'. It is easy to see that constraint functions α and β must be integrable and they should satisfy the condition

$$\int_a^b\alpha\left(x\right)dx\leq u_b-u_a\leq\int_a^b\beta\left(x\right)dx.$$

We do not use any else assumption about the constraint functions α and β.

In order to illustrate the essential features of this kind of optimization problems, we will use an elementary model given by

$$\begin{array}{l} \min_y I\left(y\right)=\int_0^1\left\{f\left(y'\left(x\right)\right)+y\left(x\right)^2\right\}dt \\ s.t. \\ y'\leq\frac{1}{2} \\ and\ y\left(0\right)=0,\ y\left(1\right)=s\,. \end{array} \tag{3}$$

where f is the fourth degree polynomial

$$f\left(t\right)=\left(1-t^2\right)^2$$

which has two global minima at ±1. Thus, we are compelled to find a curve $y\left(x\right)$ going from point $\left(0,0\right)$ to point $\left(1,s\right)$, whose derivative y' must minimize the integral $\int_0^1 f\left(y'\left(x\right)\right)dx$ by choosing values not exceeding one half.

At the same time, the same curve y should take smaller absolute values along its way in order to minimize the integral $\int_0^1 y\left(x\right)^2\, dx$. However, it is not completely clear when we can find such a curve. As instance, consider the case when $s=\frac{1}{2}$, then the straight line $y_0\left(x\right)=\frac{x}{2}$ satisfies the boundary conditions but it is not clear to us either if $I\left(y_0\right)$ is the least value among all curves $y\left(x\right)$ connecting point $(0,0)$ to point $\left(1,\frac{1}{2}\right)$ or not.

On the other hand, considering the case $s=0$, we easily see that there is no solution for the Problem 9 because we can exhibit a minimizing sequence for the functional I by choosing highly oscillatory continuous curves which decrease the value of the integral I but they never attain a minimum of I. The construction of these curves proceeds in the following form: their slopes must take values from -1 and $\frac{1}{2}$ in the relative proportions $1:2$ along the axis x. By constructing one curve in this way, it should reach the point $(1,0)$ when starting from the point $(0,0)$. In addition, by following this construction we can see that the higher the slope changes, the lesser the value of the integral I. In this way we have obtained a minimizing sequence for the functional I.

One important question about this kind of problems is about the oscillatory behavior of their minimizing sequences when they lack of minimizers. The method proposed in this paper will help us to clarify what kind of oscillatory behavior can we expect when the problem lacks of minimizers. This point has very important practical implications in several models described by variational principles like those given in the general form 7.

In order to analyze one particular non convex problem in the form 7 , we use its generalized formulation in Young measures. See [Pe1997] for a good introduction to Young measures theory. In this reference, the author uses Young measures theory for describing weak convergence in Sobolev spaces, so he can also apply it for analyzing non convex variational problems. Here we define the *generalized formulation* in Young measures of one particular derivative constrained variational problem given in the form 7, as the new optimization problem

$$\begin{array}{l}\min_\nu \widetilde{I}\left(\nu\right)=\int_a^b \int_R f\left(x,y\left(x\right),\lambda\right)\, d\mu_x\left(\lambda\right)\, dx\\ s.t.\\ support\left(\mu_x\right)\in\left[\alpha\left(x\right),\beta\left(x\right)\right]\quad for\ every\quad x\in[a,b]\\ and\quad Boundary\quad Conditions: y\left(a\right)=y_a,\quad y\left(b\right)=y_b\end{array}\tag{4}$$

which consists in determining a parametrized family of probability measures

$$\nu=\{\mu_x : a\le x\le b\}$$

that minimize the generalized functional $\widetilde{I}$. In the generalized formulation 10 the link between y and ν is given by the following restriction:

$$y'\left(x\right)=\int_R \lambda\, d\mu_x\left(\lambda\right).\tag{5}$$

Thus, we can determine every appearance of $y\left(x\right)$ in 10 by using the first moment of every parametrized measure μ_x and the boundary conditions of the particular problem.

We will show the equivalence between Problem 7 and its generalized formulation in Young measures given by the Problem 10. In fact we claim that Problem 10 has a minimizer, that means that there exists an optimal Young measure which attain a minimum for the generalized functional $\widetilde{I}$. The infimum of both problems are equal, that means that the minimum of Problem 10 equals the infimum of Problem 7. Furthermore, the supports of the optimum parametrized measures of the generalized Problem 10 clarify the existence of minimizers of the corresponding derivative constrained variational problem 7. Even if the constrained variational problem 7 does not have any minimizer within its respective Sobolev space, we can find the essential features of the oscillating behavior of their minimizing sequences by analyzing the supports of the parametrized measures inside the optimal Young measure solution of their corresponding generalized problem 10.

Now we settle the principal proposal of this work: as an alternative method for carrying out the analysis of derivative constrained variational problems with Young measures, we propose to use the algebraic moments of every parametrized moments instead of use the probability measure itself. Thus, when the integrand f of the general derivative constrained variational problem 7 has the polynomial form given in 8, the corresponding generalized problem 10 can be posed in the following way:

$$\begin{array}{l} \min_m J\left(m\right) = \int_a^b \sum_{k=0}^n c_k\left(x, y\left(x\right)\right)\, m_k\left(x\right)\, dx \\ s.t. \\ m_k\left(x\right) = \int_R \lambda^k \mu_x\left(\lambda\right) \\ support\left(\mu_x\right) \in \left[\alpha\left(x\right), \beta\left(x\right)\right] \quad for\ every \quad x \in [a,b] \\ and \quad Boundary \quad Conditions: y\left(a\right) = y_a, \quad y\left(b\right) = y_b \end{array} \tag{6}$$

which is a new optimization problem whose variables $m_k\left(x\right)$ must be constrained to be the algebraic moments of one probability measure μ_x supported on the interval $\left[\alpha\left(x\right), \beta\left(x\right)\right]$. As we explained above, the appearances of $y\left(x\right)$ in 12 can be determined from the boundary conditions and the relation 11, which takes now the simpler form:

$$y\left(x\right) = m_1\left(x\right)$$

where we have used the order-one algebraic moment of the parametrized measure μ_x.

Now we remark that Problem 12 can be described as a convex optimal control problem, because we can use classical results on moments theory for characterizing a set of values m_k as the algebraic moments of a positive measure supported on one particular interval of the real line. The most important fact about this observation is that we can estimate the optimal parametrized measures for Problem 10 by solving the Problem 12 with standard numerical procedures. In addition, all the results about the generalized formulation 10 remain true when applied to the new formulation in moments 12. In this way, we claim first that the optimal control problem 12 has a solution, second that its minimum equals the infimum of the corresponding derivative constrained problem 7. Finally, we also claim that the optimal values of the Problem 12 allow us to find the optimal parametrized measures which solve the generalized problem 12, which in turn clarify either or not the Problem 7 has minimizers within its corresponding Sobolev space. We remind that the optimal Young measure obtained from the solution of the Problem 12 describes the oscillatory behavior of the minimizing sequences of the Problem 7 when it has no minimizer.

The present paper is organized as follows. In Chapter 2 we apply Young's theory of generalized curves to the family of derivative constrained variational problems like those described in Problem 7. Then we apply Young measures theory for analyzing this kind of problems. In Chapter 3 we describe the Method of Moments for analyzing generalized problems in the form 10 and we also analyze the Problem 12 by using classical results from algebra like the famous Stieltjes and Hausdorff's Moment Problems. In Chapter 4 we analyze and solve particular examples of non convex variational problems with restrictions on the derivative. Finally, in Chapter 5 we will give some comments and remarks about this research and its links with other works and future developments. **

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problems given in the following form:

$$\begin{array}{l} \min_y I\left(y\left(x\right)\right) = \int_a^b f\left(x, y\left(x\right), y'\left(x\right)\right)\, dx \\ s.t. \\ \alpha\left(x, y\left(x\right)\right) \leq y'\left(x\right) \leq \beta\left(x, y\left(x\right)\right) \quad for\ every \quad x \in [a, b] \\ and \quad Boundary \quad Conditions: y\left(a\right) = y_a, \quad y\left(b\right) = y_b \end{array} \tag{7}$$

where the admissible functions y are supposed to belong to a proper Sobolev space according to the particular integrand f. We will study the derivative constrained variational problem 7 when the integrand f is given by the general expression

$$f\left(x, y, \lambda\right) = \sum_{k=0}^{p} c_k\left(x, y\right) \lambda^k \tag{8}$$

which means that f is an arbitrary algebraic polynomial in the derivative y'. It is remarkable that we do not use any assumption about the constraint functions α and β here.

Let us to illustrate the essential features of this kind of optimization problems by using an elementary model given as

$$\begin{array}{l} \min_y I\left(y\right) = \int_0^1 \left\{\phi\left(y'\left(x\right)\right) + y\left(x\right)^2\right\} dt \\ s.t. \\ y' \leq \frac{1}{2} \\ and\ y\left(0\right) = 0,\ y\left(1\right) = s\,. \end{array} \tag{9}$$

where ϕ is the fourth degree polynomial

$$\phi(t) = \left(1 - t^2\right)^2$$

with two global minima at points ± 1. In this case, we are compelled to find a curve $y(x)$ going from point $(0,0)$ to point $(1,s)$, whose derivative y' must minimize the integral $\int_0^1 \phi(y'(x))\, dx$ by choosing values not exceeding one half.

Simultaneously, the same curve y should take smaller absolute values along its way in order to minimize the integral $\int_0^1 y(x)^2\, dx$. However, it is not completely clear when we can find such a curve. As instance consider the case when $s = \frac{1}{2}$, in this case the straight line $y_0(x) = \frac{x}{2}$ satisfies the boundary conditions, but it is not clear if $I(y_0)$ is the least value among all curves $y(x)$ connecting the point $(0,0)$ to the point $\left(1, \frac{1}{2}\right)$.

On the other hand, if we consider the case $s = 0$ in the problem 9 we easily realize that it does not have any solution. To show this fact we must first exhibit a minimizing sequence for the functional I. This can be carried out by constructing a family of highly oscillatory continuous curves which decrease the value of the integral I without attaining a minimum for I at all. The construction of such a family of curves proceeds in a simple way: choose their slopes by taking values from -1 and $\frac{1}{2}$ by preserving the relative proportions $1:2$ along the interval $[0,1]$. As instance, after dividing the interval $[0,1]$ in $N = 3k$ subintervals uniformly, draw a poligonal line starting from $(0,0)$ with slope -1 on a selection of k subintervals, then asign a slope of $\frac{1}{2}$ to the remaining subintervals. By constructing one curve in this way, you will reach the boundary condition point $(1,0)$. From another point of view, if you follow this procedure by mixing -1 slopes and $\frac{1}{2}$ slopes, you get progressive lesser values for the integral I. This slope mixing causes the curve to oscillate, therefore the *faster* the slope transitions, the lesser the value of the integral I. In this way we have sketched a procedure for explicitely constructing minimizing sequences for the functional I, therefore the functional I lacks of minimizers because the previous construction shows that $\inf I = 0$, but it is evident that we can not find any admissible function y satisfying $I(y) = 0$.

One important question about this kind of problems is to find the way to describe the oscillatory behavior of their minimizing sequences when they lack of minimizers. The method proposed in this paper will help us to clarify what kind of oscillatory behavior can we expect when the problem lacks of minimizers. This point has very important practical implications in several models described by variational principles as those given in the form 7.

In order to analyze one particular non convex problem in the form 7, we use its generalized formulation in Young measures. Refer to [Pe1997] and [Pe2000] for an introduction to Young measures theory and its applications to variational principles. The *generalized formulation* in Young measures of the variational problem 7 is given as the following optimization problem:

$$\begin{array}{l} \min_{\nu} \widetilde{I}(\nu) = \int_a^b \int_R f(x, y(x), \lambda)\, d\mu_x(\lambda)\, dx \\ s.t. \\ support(\mu_x) \in [\alpha(x, y(x)), \beta(x, y(x))] \quad for\ every \quad x \in [a, b] \\ and \quad Boundary \quad Conditions: y(a) = y_a, \quad y(b) = y_b \end{array} \tag{10}$$

in which we must find a parametrized family of probability measures

$$\nu = \{\mu_x : a \leq x \leq b\}$$

which must minimize the generalized functional $\widetilde{I}$. In this formulation the link between y and ν is given by the restriction:

$$y'(x) = \int_R \lambda\, d\mu_x(\lambda). \tag{11}$$

Thus, we can determine every appearance of $y(x)$ in 10 by using the first moment of every parametrized measure μ_x and the boundary conditions imposed in the problem.

We will show certain sort of equivalence between problem 7 and the corresponding formulation in Young measures given by the generalized problem 10. Indeed, we will see that problem 10 always has a minimizer, and we also see that its minimum equals the infimum of problem 7. This is a well known *relaxation* result. It is worth noticing here, that the supports of the optimal parametrized measures for problem 10 determine the existence of minimizers of the corresponding variational problem 7. In fact, they can describe the oscillatory behavior inherent in every minimizing sequence of the functional I. Therefore, the abscense of oscillations means the existence of minimizers and conversely the evidence for oscillatory behavior means the lack of minimizers.

Now we present the major proposal of this work: in order to carry out the analysis of derivative constrained variational problems 7 when the integrand is given by a polynomial like 8, we must represent every parametrized measure in 10 by using its algebraic moments. Thus, every generalized problem 10 can be transformed into the following optimization problem:

$$\begin{array}{l} \min_m J(m) = \int_a^b \sum_{k=0}^p c_k(x, y(x))\, m_k(x)\, dx \\ s.t. \\ m_k(x) = \int_R \lambda^k \mu_x(\lambda) \quad for \quad k = 0, \ldots, n \\ support(\mu_x) \in [\alpha(x, y(x)), \beta(x, y(x))] \quad for\ every \quad x \in [a, b] \\ and \quad Boundary \quad Conditions: y(a) = y_a, \quad y(b) = y_b \end{array} \tag{12}$$

whose variables $m_k(x)$ are the algebraic moments of the parametrized measure μ_x, which in turn must be supported on the interval $[\alpha(x, y(x)), \beta(x, y(x))]$. As we explained above, the appearances of $y(x)$ in 12 can be determined from the boundary conditions and the relation 11, which takes here the simpler form:

$$y(x) = m_1(x)$$

where we have used the first moment of the parametrized measure μ_x.

We stressed that problem 12 can be described as an optimal control problem, because we can use classical results on moments theory for characterizing a set of values m_k as the algebraic moments of any positive measure supported on a particular interval of the real line. The most important fact about this observation is that we can estimate optimal parametrized measures for generalized problems 10 by solving the corresponding moments control problem 12. Thus we are exhibiting a practical way for clarifying either or not the original problem 7 has minimizers and for describing the oscillatory behavior in its minimizing sequences too.

The present paper is organized as follows. In Chapter 2 we apply Young measures theory for analyzing the family of derivative constrained variational problems 7. In Chapter 3 we describe the Method of Moments for analyzing generalized problems in the form 10 and we also analyze the Problem 12 by using classical results from algebra like the famous Stieltjes and Hausdorff's Moment Problems. In Chapter 4 we analyze and solve particular examples of non convex variational problems with restrictions on the derivative. Finally, in Chapter 5 we will give some comments and remarks about this research and its links with other works and future developments.

2 General Analysis of the Problem

In order to analyze the variational problem 7, we must transform it into a standard variational problem by including its derivative constraints into the definition of a new integrand $\overline{f}$ defined as follows:

$$\overline{f}\,(x,y,\lambda)=\begin{cases}\sum_{k=0}^{p} c_k\,(x,y)\,\lambda^k & if \quad (x,y,\lambda)\in\mathcal{A}\\ \infty & otherwise\end{cases}$$

where $\mathcal{A}$ is the set of values (x,y,λ) satisfying the inequalities in 7, i.e.

$$\mathcal{A}=\left\{(x,y,\lambda)\in R^3:\alpha\,(x,y)\le\lambda\le\beta\,(x,y)\quad where\quad x\in[a,b]\right\}.$$

In this way, the problem 7 takes the form of the following variational problem:

$$\begin{array}{l}\min_y \overline{I}\,(y\,(x))=\int_a^b\overline{f}\,(x,y\,(x)\,,y'\,(x))\;dx\\ under\quad Boundary\quad Conditions: y\,(a)=y_a,\quad y\,(b)=y_b\end{array}\tag{13}$$

whose functional $\overline{I}$ is defined in terms of the integrand function $\overline{f}$.

In order to determine the existence of minimizers for this kind of problems we must specify a particular space as the overall set of admissible functions. Since we explicitely use the derivative of every admissible function to evaluate the functional $\overline{I}$ in 13, we may use the one dimensional Sobolev space $H^{1,p}\,(a,b)$ as the family of admissible functions where p is precissely the degree of the polynomial in 8. We warn that we should consider bounded constraint functions α and β in order to avoid incongruences with the admissible space choosen.

After determining the admissible set for the particular variational problem we are interested in, we focus on the existence of minimizers for this problem.

To solve this question we should use the *Direct Method* from the modern theory of the Calculus of Variations. If we assume that $\overline{I}$ is a coercive, weakly semi-continuous functional in the space of admissible functions, then we can conclude that $\overline{I}$ has at least one minimizer in

$$H_0^{1,p}(a,b) + y_a + (x-a)\frac{(y_b - y_a)}{b-a} \tag{14}$$

because the choosen space of admisible functions is a reflexive Banach space when $p > 1$. But this is just the drawback: we can not guaranty weak inferior semi-continuity for the functional I because the integrand f does not have necessarily a convex dependence on the derivative variable. A deeper analysis of the implications of the convexity of the integrand function f upon the weak semicontinuity of the functional I can be found in [Da1999], [Jo1998] and [Bu1998].

In the particular case in which f and α are convex and β is concave, we can easily analyze the variational problem 13 by the Direct Method of the Calculus of Variations. Indeed, if the functional $\overline{I}$ takes a finite value on some admissible function and $\overline{f}$ is coercive, i.e.

$$f(x,y,\lambda) \geq g(x) + c\,|\lambda|^p \tag{15}$$

for some integrable function g in $[a,b]$ and some positive constant c in the region defined by $\mathcal{A}$, then the existence of minimizers for $\overline{I}$ is guaranteed within the space of admissible functions 14. See [Da1999, Theorem 4.1, pag 82.].

Since formulation 13 describes a general family of nonconvex variational problems, we can apply a particular relaxation technique involving Young measures for analyzing them. This technique consists in formulating every non convex problem 13 as a new optimization problem, defined in parametrized measures, with the following form:

$$\begin{array}{l} \min_{\nu\in Y} \widetilde{I}(\nu) = \int_a^b \left(\int_R \overline{f}(x, y(x), \lambda)\; d\mu_x(\lambda) \right) dx \\ s.t. \\ y'(x) = \int_R \lambda \, d\mu_x(\lambda) \\ and\ Boundary\quad Conditions : y(a) = y_a, \quad y(b) = y_b. \end{array} \tag{16}$$

In this new optimization problem the objective function is a *generalized* functional $\widetilde{I}$ defined in a vast space of parametrized probability measures Y. Each memeber ν in Y is a whole family of probability measures μ_x indexed by a point x in $[a,b]$, i.e.

$$v = \{\mu_x : a \leq x \leq b\} \quad for\ every\ \nu \in Y.$$

This amounts to replace the admissible function $u : [a,b] \to R$ by taking a new sort of function $\nu : [a,b] \to \mathcal{P}$ whose values are probability distributions. It is customary to refer every parametrized family ν as a *Young measure.* The essential feature of Young measures in functional analysis is their ability to describe weak convergence in Lebesgue and Sobolev spaces, then they become a fundamental tool for proving existence of minimizers for variational problems. See [Pe1997] and [Pe2000] for a thorough exposition on the applications of Young measures to variational problems.

We shortly present the basic ideas behind the theory of Young measures as applied to variational problems. In order to prove the existence of minimizers for a functional like

$$I(u) = \int_{\Omega} f(u(x))\, dx \tag{17}$$

where f is a continuous function with positive polynomial growth toward the infinities, we can use the Direct Method provided the functional I be weakly inferior semicontinuous in a reflexive Banach space X and coercive, i.e.

$$I(u) \to \infty \quad \text{whenever} \quad \|u\| \to \infty.$$

Under these assumptions, every minimizing sequence $\{u_n : n \in N\}$ has a weakly convergent subsequence in X, i.e.

$$u_{n_k} \rightharpoonup \overline{u}$$

for $\overline{u} \in X$. We easily see that $\overline{u}$ is a minimizer for I, just notice that

$$\inf_X I \leq I(\overline{u}) \leq \liminf_k I(u_{n_k}) = \inf_X I.$$

We immediately see that $f \circ \overline{u}$ is the weak limit of $f \circ u_{n_k}$.

However, in abscense of the weak semicontinuous hypothesis we can not apply this argument any more. Fortunately, we can analyze the weak limit of $f \circ u_n$ for every minimizing sequence as this is precisely the role of Young measures in analysis. In short, for every sequence of admissible functions $\{u_n : n \in N\}$ we characterize the weak limit of the sequence $f \circ u_n$ as

$$\phi(x) = \int_R f(\lambda)\, d\mu_x(\lambda) \tag{18}$$

where $\nu = \{\mu_x : a \leq x \leq b\}$ is the Youg measure linked with the sequence $\{u_n : n \in N\}$.

Under mild hypothesis, every sequence of admissible functions has a particular Young measure which provides the weak limit of the resulting sequence when its terms are composed with a particular continuous function f with polynomial growth, as has been just expressed in 18. Therefore, two sequences share the same Young measure as far as they share all their weak limits after composition with continuos functions. Thus, every Young measure represents the weak limits of a huge family of function sequences when they are composed with continuous functions. See [Pe1997].

Given an arbitrary sequence of admissible functions $\{u_n : n \in N\}$ whose Young measure is ν, the weak limit of the sequence $f \circ u_n$ can be obtained by using the integral in 18. Therefore, we can determine minimizing sequences for the functional I by solving the following generalized variational problem

$$\min_{v\in Y} \widetilde{I}(\nu) = \int_{\Omega}\int_{R} f(\lambda)\,\mu_x(\lambda)\, dx. \tag{19}$$

It is worth noticing that we elude convexity assumptions as we are using Young measures for describing weak limits of sequences like $f \circ u_n$. The only assumptions on f are continuity and polynomial growth in the infinities. This is a remarkable achivement as has been shown that the Direct Method fails in absence of weak semicontinuity which in turns comes from convexity in f. See [Da1999] and [Pe1997].

By assuming polynomial growth in the function f and by using convex analysis tools, we can show that optimal parametrized measures for problem 19 are supported on many finite points. In the special case in which every optimal parametrized measure is supported on a single point, i.e.

$$\mu_x^* = \delta_{g(x)}\ for\ every\ x \in [a,b]$$

where δ_t represents the trivial distribution supported on t (also called *Dirac measure)*, we conclude that $u^*(x) = g(x)$ is a minimizer for I. This turns out to be true from the fact that

$$\phi(x) = \int_R f(\lambda)\, d\mu_x^*(\lambda) = f(g(x))$$

is the weak limit of $f \circ u_n$ whenever u_n be a minimizing sequence for I. In general, the optimal parametrized measures μ_x^* are supported on finite sets containing a fixed maximum number of points, let us to say two, which is true in one-dimensional cases. Thus we obtain the general expression

$$\mu_x^* = p_1(x)\,\delta_{g_1(x)} + p_2(x)\,\delta_{g_2(x)} \tag{20}$$

where

$$p_1(x) + p_2(x) = 1, \quad p_i(x) \geq 0$$

for every $x \in [a,b]$ and $i = 1,2$.

Every probability measure in 20 gives us the information about the strategy that we should impose in the function u in order to minimize the functional I. As we explained above, the function

$$\phi(x) = \int_R f(\lambda)\, d\mu_x^*(\lambda) = p_1(x)\, f(g_1(x)) + p_2(x)\, f(g_2(x)) \tag{21}$$

provides the weak limit of any sequence $f \circ u_n$ whenever u_n is a minimizing sequence for I. Therefore, the expression 21 gives us a method for revealing the kind of oscillatory behavior we should observe in every minimizing sequence of I. In order to decrease the value of the integral I in 18, the function u should take values from $g_1(x)$ and $g_2(x)$ by using the relative proportions $p_1(x)$ and

$p_2(x)$ inside an infinitesimal neighbourhood of the point x. When this analysis is applied to the derivative of the function, we can observe several kinds of oscillatory phenomena in the minimizing sequences of the functional I. This analysis has been presented in [Pe1997] for the special case involving Sobolev spaces.

Now we apply Young measures theory to the variational problem 13 we are interested in. By assuming that the integrand f is bounded in the form expressed in 15, we can apply Poincare's inequality to conclude that every minimizing sequence for the functional $\overline{I}$ in 13 must be bounded in the Sobolev space $H^{1,p}(a,b)$. Then, the generalized problem 16 admits a minimizer and we have the next ralaxation result:

$$\min_{\nu \in Y} \widetilde{I} = \inf_{X} \overline{I}$$

where X is the admissible functions space given in 14. See [Pe1997, Theorem 4.4, page 67].

It is very important for our work noticing that the optimal Young measure

$$\nu^* = \{\mu_x^* : a \leq x \leq b\}$$

for the generalized functional $\widetilde{I}$ in 16 satisfies the following properties:

- $\overline{f}_c(x, y(x), \lambda) = \int_R \overline{f}(x, y(x), \lambda)\, \mu_x^*(\lambda)\, dx$ where

$$y(x) = y_a + \int_a^x \int_R \lambda \mu_x(\lambda)\, dx$$

  and $\overline{f}_c$ stands for the convex envelope of the function $\lambda \to f(x, y, \lambda)$ while keeping fix x and y. See [Pe1997, Corollary 4.6, page 68.]

- The support of every optimal parametrized measure μ_x^* must be contained in the set of points λ satisfying $\overline{f}_c(x, y(x), \lambda) = \overline{f}(x, y(x), \lambda)$ for every point $x \in [a, b]$.[Pe1997, Corollary 4.7, page 68]

- The variational problem 13 has a minimizer $y^*(x) = y_a + \int_a^x g(s)\, ds$ if and only if the optimal Young measure for 16 is composed of Dirac measures in the form

$$\mu_x^* = \delta_{g(x)}.$$

  [Pe1997, Proposition 6.12, page 111].

Now we can see easily that our generalized problem 16 can be posed as the problem 12 because no point in the support of any parametrized measure can be within the region where $\overline{f}$ have infinite values. This would cause I to take infinite values. In the sequel we analyze problem 12 which is a generalized variational problem in which the factible parametrized measures are supported in the interval defined by the restrictions α and β of the original problem 13.

3 Examples

3.1 Example 1

We consider the following variational problem:

$$\begin{aligned} \min \int_0^1 &(1-u'(t)^2)^2 + u(t)^2 dt \\ s.t. u(0) &= a \\ u(1) &= b \\ u'(t) &\le \alpha \end{aligned}$$

We can write the relaxed problem as:

$$\begin{aligned} \min \int_0^1 &\left(1 - 2m_2 + m_4 + \int_0^t u(s)ds\right) dt \\ s.t. u(0) &= a \\ u(1) &= b \\ &\begin{bmatrix} 1 & m_1 & m_2 \\ m_1 & m_2 & m_3 \\ m_2 & m_3 & m_4 \end{bmatrix} \ge 0 \\ &\begin{bmatrix} \alpha - m_1 & \alpha m_1 - m_2 \\ \alpha m_1 - m_2 & \alpha m_2 - m_3 \end{bmatrix} \ge 0 \end{aligned}$$

3.1.1 Case 1

This program is solved for $a = b = 0$ and for $\alpha = 0.5$ and we have that the measure is:

$$\mu^* = 0.3456\delta_{-0.9485} + 0.65544\delta_{0.5}$$

The figure 1 shows the oscillations presented in $u(t)$

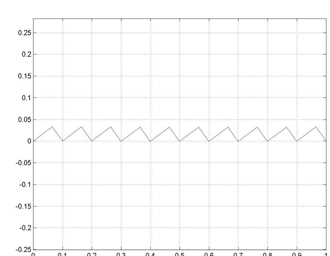

Figure 1: u(t) for the example 1

3.1.2 Case 2

Now we do the same procedure for $a = 0$, $b = 1$ and $\alpha = 0.5$. We express the measures as:

$$\mu^*(t) = \begin{cases} 0.3333\delta_{-0.9469} + 0.6666\delta_{0.5} & if \quad t \in [0, 0.4) \\ 0.2462\delta_{-0.9469} + 0.7538\delta_{0.5} & if \quad t \in [0.4, 1] \end{cases}$$

The figure 2 shows the oscillations presented in $u(t)$ for this case

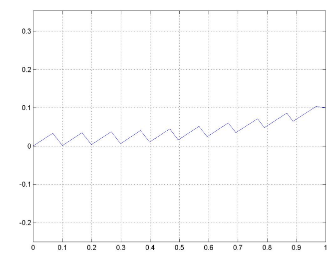

Figure 2: u(t) for the example 1, case 2

3.1.3 Case 3

For the point $a = 0$, $b = 0.2$, y $\alpha = 0.5$ we have the measure:

$$\mu^* = 0.2\delta_{-0.9620} + 0.8\delta_{0.5}$$

the figure 3 shows the results for this case. Now, we take a point in the convex

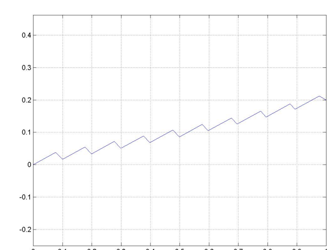

Figure 3: u(t) for the example 1, case 3

zone, $a = 0$, $b = 0.5$ and $\alpha = 0.5$, the measure is:

$$\mu^* = \delta_1$$

3.1.4 Case 4

For the next set of point, we take $a = 0.5$, $b = 0$, $\alpha = 0.5$, we have the following measures variables in time in the table 1: The figure 4 show the results for u^*

t	μ^*
0.1	$1\delta_{0.9804}$
0.2	$1\delta_{0.9617}$
0.3	$1\delta_{0.9297}$
0.4	$0.7869\delta_{0.9035} + 0.2131\delta_{0.5}$
0.5	$0.6718\delta_{0.9252} + 0.3282\delta_{0.5}$
0.6	$0.5844\delta_{0.9289} + 0.4156\delta_{0.5}$
0.7	$0.5127\delta_{0.9342} + 0.4873\delta_{0.5}$
0.8	$0.4892\delta_{0.9540} + 0.5108\delta_{0.5}$
0.9	$0.4507\delta_{0.9626} + 0.5493\delta_{0.5}$
1	$0.4250\delta_{-0.972} + 0.5750\delta_{0.5}$

Table 1: Table of measure

in this case.

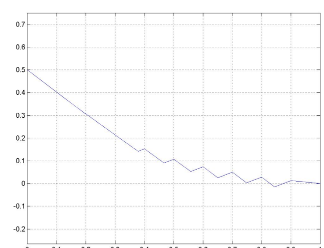

Figure 4: u(t) for the example 1, case 4

3.1.5 Case 5

Now, we take as final point $b = 0.3$, $a = 0.5$ and $\alpha = 0.5$. The measure is shown in the table 2: We show in the figure 5 the result for $u^*(t)$,

3.1.6 Case 6

We take the obvious case, when $a = 0$, $b = 0$ and $\alpha = 0$. Here we have a case that the problem is interesting just for proving the results. We got here a measure:

$$\mu^* = \delta_0$$

And we have the results shown in the figure 6.

t	μ^*
0.1	$1.0000\delta_{-0.9804}$
0.2	$1.0000\delta_{-0.9617}$
0.3	$1.0000\delta_{-0.9297}$
0.4	$0.7869\delta_{-0.9035} + 0.5\delta_{0.2131}$
0.5	$0.6718\delta_{-0.9252} + 0.5\delta_{0.3282}$
0.6	$0.5844\delta_{-0.9289} + 0.5\delta_{0.4156}$
0.7	$0.5127\delta_{-0.9342} + 0.5\delta_{0.4873}$
0.8	$0.4892\delta_{-0.9540} + 0.5\delta_{0.5108}$
0.9	$0.4507\delta_{-0.9626} + 0.5\delta_{0.5493}$
1	$0.4250\delta_{-0.9720} + 0.5\delta_{0.5750}$

Table 2: Table of measure

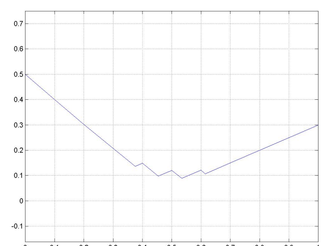

Figure 5: u(t) for the example 1, case 5

3.1.7 Case 7

We take the case that the pendant must be greater than zero. So we take $a = 0$, $b = 0.5$ and $\alpha = 0$. We have here a measure:

$$\mu^* = 0.5\delta_0 + 0.5\delta_1$$

The oscillations can be easily seen in the figure 7.

3.2 Example 2

Now we change a little bit the variational problem:

$$\begin{aligned} &\min \int_0^1 (1 - u'(t)^2)^2 + u(t)^2 dt \\ &s.t. u(0) = a \\ &\quad u(1) = b \\ &\quad u'(t) \geq \alpha \end{aligned}$$

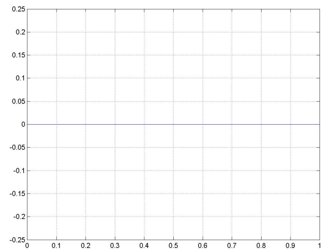

Figure 6: u(t) for the example 1, case 6

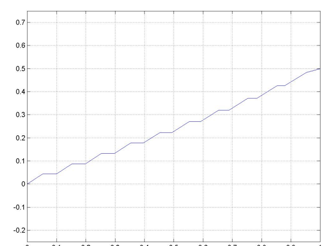

Figure 7: u(t) for the example 1, case 7

The relaxed problem take the form:

$$
\begin{aligned}
&\min \int_0^1 \left(1 - 2m_2 + m_4 + \int_0^t u(s)ds\right) dt \\
&s.t. u(0) = a \\
&\quad u(1) = b \\
&\quad \begin{bmatrix} 1 & m_1 & m_2 \\ m_1 & m_2 & m_3 \\ m_2 & m_3 & m_4 \end{bmatrix} \geq 0 \\
&\quad \begin{bmatrix} m_1 - \alpha & m_2 - \alpha m_1 \\ m_2 - \alpha m_1 & m_3 - \alpha m_2 \end{bmatrix} \geq 0
\end{aligned}
$$

3.2.1 Case 1

We take the case when $a = 0$, $b = 0$ and $\alpha = -0.1$. The measure in this case is:

$$\mu^* = 0.9\delta_{-0.1} + 0.1\delta_{0.8484}$$

And the result for this case is the oscillations that are shown in the figura 8.

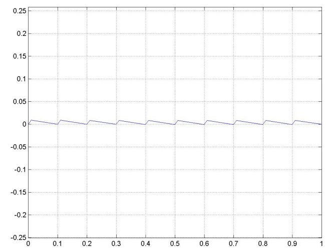

Figure 8: u(t) for the example 2, case 1

3.2.2 Case 2

We take the case $a = 0$, $b = 0.2$, $\alpha = -0.1$, and we have the measure shown in the table **??** And we have here the figure 9.

t	μ^*
0.1	$0.7137\delta_{-0.1000} + 0.2863\delta_{0.8665}$
0.2	$0.7124\delta_{-0.1000} + 0.2876\delta_{0.8654}$
0.3	$0.7099\delta_{-0.1000} + 0.2901\delta_{0.8633}$
0.4	$0.7060\delta_{-0.1000} + 0.2940\delta_{0.8601}$
0.5	$0.7000\delta_{-0.1000} + 0.3000\delta_{0.8560}$
0.6	$0.6896\delta_{-0.1000} + 0.3104\delta_{0.8510}$
0.7	$0.6769\delta_{-0.1000} + 0.3231\delta_{0.8451}$
0.8	$0.6617\delta_{-0.1000} + 0.3383\delta_{0.8382}$
0.9	$0.6437\delta_{-0.1000} + 0.3563\delta_{0.8303}$
1	$0.6226\delta_{-0.1000} + 0.3774\delta_{0.8215}$

Table 3: Table of measure

3.2.3 Case 3

Now we take $\alpha = -x/2$ for this problem, $a = 0$, $b = 0$. The measure will be given for the values in the table 4 The figure 10 shows the result for $u^*(t)$.

3.3 Example 3

We take a different polynomial for the functional. In this case we will solve the variational problem:

$$\begin{aligned} \min \int_0^1 0.5 + u'(t)^4 - 0.8u'(t)^3 - 0.5u'(t)^2 + 0.5u'(t) + u(t)^2 dt \\ s.t. u(0) = a \\ u(1) = b \\ u'(t) \geq \alpha \end{aligned}$$

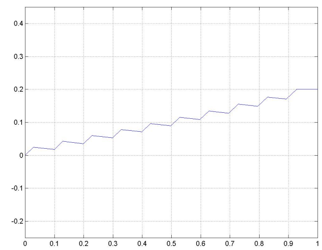

Figure 9: u(t) for the example 2, case 2

t	μ^*
0.1	$1.0000\delta_{0.8294} + 0\delta_0$
0.2	$0.2138\delta_{-0.1000} + 0.7862\delta_{0.8337}$
0.3	$0.4846\delta_{-0.1500} + 0.5154\delta_{0.8640}$
0.4	$0.7219\delta_{-0.2000} + 0.2781\delta_{0.9006}$
0.5	$0.8577\delta_{-0.2500} + 0.1423\delta_{0.9052}$
0.6	$0.8900\delta_{-0.3002} + 0.1100\delta_{0.9108}$
0.7	$\delta_{-0.3523}$
0.8	$\delta_{-0.4020}$
0.9	$\delta_{-0.4500}$
1	$\delta_{-0.5000}$

Table 4: Table of measure

This problem in moments can be taken as:

$$\min \int_0^1 \left(0.5 + m_4 - 0.8m_3 - 0.5m_2 + 0.5m_1 + \left(\int_0^t u(s)ds\right)^2\right) dt$$

$$s.t. u(0) = a$$

$$u(1) = b$$

$$\begin{bmatrix} 1 & m_1 & m_2 \\ m_1 & m_2 & m_3 \\ m_2 & m_3 & m_4 \end{bmatrix} \geq 0$$

$$\begin{bmatrix} m_1 - \alpha & m_2 - \alpha m_1 \\ m_2 - \alpha m_1 & m_3 - \alpha m_2 \end{bmatrix} \geq 0$$

3.3.1 Case 1

We take the above problem with the points $a = 0$, $b = 0.1$, and $\alpha = 0$. The table 5 shows the measure after solving this problem. The figure 11 shows the graphic for this case.

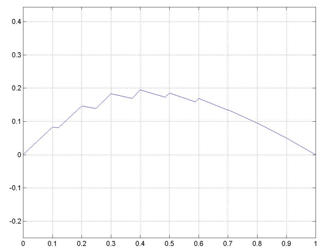

Figure 10: u(t) for the example 2, case 3

t	μ^*
0.1	$0.8752\delta_{-0.0013} + 0.1248\delta_{0.7458}$
0.2	$0.8744\delta_{-0.0013} + 0.1256\delta_{0.7453}$
0.3	$0.8730\delta_{-0.0013} + 0.1270\delta_{0.7441}$
0.4	$0.8708\delta_{-0.0013} + 0.1292\delta_{0.7424}$
0.5	$0.8678\delta_{-0.0013} + 0.1322\delta_{0.7401}$
0.6	$0.8640\delta_{-0.0013} + 0.1360\delta_{0.7371}$
0.7	$0.8593\delta_{-0.0013} + 0.1407\delta_{0.7335}$
0.8	$0.8537\delta_{-0.0013} + 0.1463\delta_{0.7292}$
0.9	$0.8471\delta_{-0.0013} + 0.1529\delta_{0.7242}$
1	$0.8393\delta_{-0.0013} + 0.1607\delta_{0.7184}$

Table 5: Table of measure

3.3.2 Case 2

We are interesting for this problem in the case $a = -0.5$, $b = 0$, $\alpha = 0$. In the table 6, we show the measure. The figure 12 shows the results in oscillations.

3.3.3 Case 3

We take this problem in the case $a = -0.5$, $b = 0$, $\alpha = x/2$. In the table 7, we show the measure. The figure 13 shows the results in oscillations.

3.4 Example 4

Now, we take the variational problem as follows:

$$
\begin{aligned}
&\min \int_0^1 ((1 - u'(t)^2)^2 + (u(t) - g(t))^2)dt \\
&s.t. u(0) = a \\
&\quad u(1) = b \\
&\quad \alpha \le u'(t) \ge \beta
\end{aligned}
$$

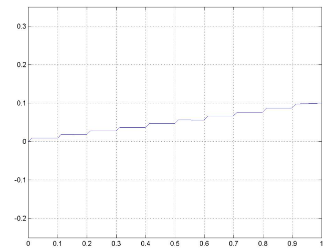

Figure 11: u(t) for the example 3, case 1

t	μ^*
0.1	$1.0000\delta_{0.7963}$
0.2	$1.0000\delta_{0.7792}$
0.3	$1.0000\delta_{0.7332}$
0.4	$0.3837\delta_{-0.0006} + 0.6163\delta_{0.7424}$
0.5	$0.3210\delta_{-0.0002} + 0.6790\delta_{0.7401}$
0.6	$0.4972\delta_{-0.0000} + 0.5028\delta_{0.7371}$
0.7	$0.5849\delta_{-0.0000} + 0.4151\delta_{0.7335}$
0.8	$0.6370\delta_{-0.0000} + 0.3630\delta_{0.7292}$
0.9	$0.5995\delta_{-0.0001} + 0.4005\delta_{0.7242}$
1	$0.6042\delta_{-0.0001} + 0.3958\delta_{0.7184}$

Table 6: Table of measure

The relaxed problem can be written as:

$$
\begin{aligned}
&\min \int_0^1 \left(1 - 2m_2 + m_4 \left(\int_0^t u(s)ds - g(t) \right)^2 \right) dt \\
&s.t. u(0) = a \\
&\quad u(1) = b \\
&\quad \begin{bmatrix} 1 & m_1 & m_2 \\ m_1 & m_2 & m_3 \\ m_2 & m_3 & m_4 \end{bmatrix} \geq 0 \\
&\quad \begin{bmatrix} \alpha\beta + (\alpha+\beta)m_1 - m_2 & -\alpha\beta m_1 + (\alpha+\beta)m_2 + m_3 \\ \alpha\beta m_1 + (\alpha+\beta)m_2 - m_3 & -\alpha\beta m_2 + (\alpha+\beta)m_3 - m_4 \end{bmatrix} \geq 0
\end{aligned}
$$

3.4.1 Case 1

We take the first case when $a = 0$, $b = 0.2$, $\alpha = -0.5$, $\beta = 0.5$, $g(t) = 0$ for 10 points. The table 8 shows the measure we have for this case. The results here are summarized in the figure 14.

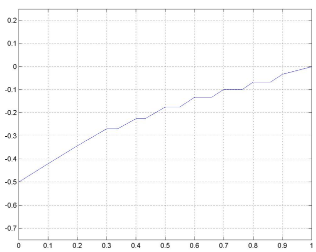

Figure 12: u(t) for the example 3, case 2

t	μ^*
0.1	$1.0000\delta_{0.7109}$
0.2	$1.0000\delta_{0.6605}$
0.3	$1.0000\delta_{0.7942}$
0.4	$0.8919\delta_{0.8198} + 0.1081\delta_{1.3221}$
0.5	$0.1234\delta_{-0.2500} + 0.8766\delta_{0.7678}$
0.6	$0.2404\delta_{-0.3002} + 0.7596\delta_{0.7647}$
0.7	$0.3830\delta_{-0.3501} + 0.6170\delta_{0.7801}$
0.8	$0.4654\delta_{-0.3961} + 0.5346\delta_{0.7878}$
0.9	$0.5411\delta_{-0.4281} + 0.4589\delta_{0.7908}$
1	$0.5596\delta_{-0.4562} + 0.4404\delta_{0.7932}$

Table 7: Table of measure

3.4.2 Case 2

We take $a = 0$, $b = 0.1$, $\alpha = -0.5$, $\beta = 0.5$, and $g(t) = t^2$. We have here the results show in the table 9 for the measure. The figure 15 shows the results for $u^*(t)$

3.4.3 Case 3

Now we take $a = 0.5$, $b = 0.2$, $\alpha = -0.5$, $\beta = 0.5$, and $g(t) = t/4$. We have the measure shown in the table 10. The result of the variational program is shown in the figure 16

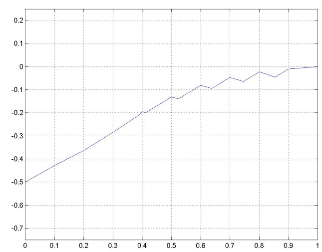

Figure 13: u(t) for the example 3, case 3

t	μ^*
0.1	$0.3989\delta_{-0.3018} + 0.6011\delta_{0.5}$
0.2	$0.3989\delta_{-0.3055} + 0.6011\delta_{0.5}$
0.3	$0.4241\delta_{-0.2529} + 0.5759\delta_{0.5}$
0.4	$0.4139\delta_{-0.2595} + 0.5861\delta_{0.5}$
0.5	$0.4065\delta_{-0.2722} + 0.5935\delta_{0.5}$
0.6	$0.4075\delta_{-0.2521} + 0.5925\delta_{0.5}$
0.7	$0.3865\delta_{-0.2666} + 0.6135\delta_{0.5}$
0.8	$0.3624\delta_{-0.2846} + 0.6376\delta_{0.5}$
0.9	$0.3352\delta_{-0.3072} + 0.6648\delta_{0.5}$
1	$0.3132\delta_{-0.3154} + 0.6868\delta_{0.5}$

Table 8: Table of measure

3.5 Example 5

We take the variational problem:

$$\begin{aligned} &\min \int_0^1 0.5 + u'(t)^4 - 0.8u'(t)^3 - 0.5u'(t)^2 + 0.5u'(t) + u(t)^2 dt \\ &s.t. u(0) = a \\ &\quad u(1) = b \\ &\quad \alpha \geq u'(t) \geq \beta \end{aligned}$$

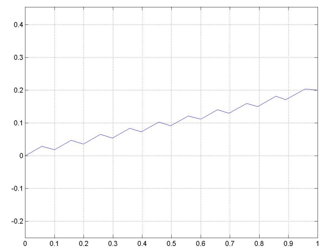

Figure 14: u(t) for the example 4, case 1

t	μ^*
0.1	$0.1450\delta_{-0.5} + 0.8550\delta_{0.5007}$
0.2	$0.1398\delta_{-0.5} + 0.8602\delta_{0.5007}$
0.3	$0.1403\delta_{-0.5} + 0.8597\delta_{0.5007}$
0.4	$0.1578\delta_{-0.5} + 0.8422\delta_{0.5010}$
0.5	$0.2037\delta_{-0.5} + 0.7963\delta_{0.5018}$
0.6	$0.2899\delta_{-0.5} + 0.7101\delta_{0.5036}$
0.7	$0.4288\delta_{-0.5} + 0.5712\delta_{0.5076}$
0.8	$0.6318\delta_{-0.5} + 0.3682\delta_{0.5161}$
0.9	$0.9133\delta_{-0.5} + 0.0867\delta_{0.5895}$
1	$1.0000\delta_{-0.5}$

Table 9: Table of measure

This problem in moments can be taken as:

$$
\begin{aligned}
&\min \int_0^1 \left(0.5 + m_4 - 0.8m_3 - 0.5m_2 + 0.5m_1 + \left(\int_0^t u(s)ds\right)^2\right) dt \\
&s.t. u(0) = a \\
&\quad u(1) = b \\
&\quad \begin{bmatrix} 1 & m_1 & m_2 \\ m_1 & m_2 & m_3 \\ m_2 & m_3 & m_4 \end{bmatrix} \geq 0 \\
&\begin{bmatrix} \alpha\beta + (\alpha+\beta)m_1 - m_2 & -\alpha\beta m_1 + (\alpha+\beta)m_2 + m_3 \\ \alpha\beta m_1 + (\alpha+\beta)m_2 - m_3 & -\alpha\beta m_2 + (\alpha+\beta)m_3 - m_4 \end{bmatrix} \geq 0
\end{aligned}
$$

3.5.1 Case 1

The first case we analyse is $a = 0$, $b = 0.2$, $\alpha = -0.5$, $\beta = 0.5$ y $g(t) = 0$. The table 11 The result of the variational program is shown in the figure 17

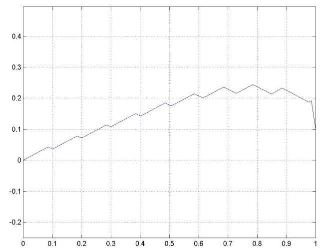

Figure 15: u(t) for the example 4, case 2

t	μ^*
0.1	$1.0000\delta_{-0.5}+$
0.2	$1.0000\delta_{-0.5}+$
0.3	$0.9611\delta_{-0.5} + 0.0389\delta_{0.4363}$
0.4	$0.8715\delta_{-0.5} + 0.1285\delta_{0.4895}$
0.5	$0.7942\delta_{-0.5} + 0.2058\delta_{0.4906}$
0.6	$0.7323\delta_{-0.5} + 0.2677\delta_{0.4908}$
0.7	$0.6856\delta_{-0.5} + 0.3144\delta_{0.4914}$
0.8	$0.6563\delta_{-0.5} + 0.3437\delta_{0.4959}$
0.9	$0.6377\delta_{-0.5} + 0.3623\delta_{0.4964}$
1	$0.6321\delta_{-0.5}0.3679\delta_{0.4966}$

Table 10: Table of measure

3.5.2 Case 2

We take $a = 0.1$, $b = 0$, $\alpha = -0.5$, $\beta = 0.5$ and $p(t) = t/4$. The table 12 shows the values we got for the measure. The result of the variational program is shown in the figure 18

3.6 Example 6

Now we take a polynomial with 6 degree. The problem is to find the minimum that:

$$\begin{aligned} &\min \int_0^1 (3.5u'(t)^2 - 4u'(t)^4 + u'(t)^6)^2 + u(t)^2 dt \\ &s.t. u(0) = a \\ &\quad u(1) = b \\ &\quad u'(t) \leq \alpha \end{aligned}$$

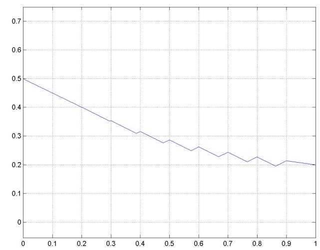

Figure 16: u(t) for the example 4, case 3

t	μ^*
0.1	$0.5508\delta_{-0.3786} + 0.4492\delta_{0.5003}$
0.2	$0.5289\delta_{-0.3814} + 0.4711\delta_{0.5002}$
0.3	$0.4912\delta_{-0.3851} + 0.5088\delta_{0.5001}$
0.4	$0.4441\delta_{-0.3849} + 0.5559\delta_{0.5003}$
0.5	$0.3834\delta_{-0.3778} + 0.6166\delta_{0.5004}$
0.6	$0.4210\delta_{-0.3852} + 0.5790\delta_{0.5002}$
0.7	$0.2754\delta_{-0.3783} + 0.7246\delta_{0.5003}$
0.8	$0.1183\delta_{-0.4212} + 0.8817\delta_{0.5009}$
0.9	$1.0000\delta_{0.5000}$
1	$0.1868\delta_{-0.3775}0.8132\delta_{0.5000}$

Table 11: Table of measure

We can write the relaxed problem as:

$$
\begin{aligned}
&\min \int_0^1 \left(3.5m_2 - 4m_4 + m_6 + \int_0^t u(s)ds\right) dt \\
&s.t. u(0) = a \\
&\quad u(1) = b \\
&\begin{bmatrix} 1 & m_1 & m_2 & m_3 \\ m_1 & m_2 & m_3 & m_4 \\ m_2 & m_3 & m_4 & m_5 \\ m_3 & m_4 & m_5 & m_6 \end{bmatrix} \geq 0 \\
&\begin{bmatrix} \alpha - m_1 & \alpha m_1 - m_2 & \alpha m_2 - m_3 \\ \alpha m_1 - m_2 & \alpha m_2 - m_3 & \alpha m_3 - m_4 \\ \alpha m_2 - m_3 & \alpha m_3 - m_4 & \alpha m_4 - m_5 \end{bmatrix} \geq 0
\end{aligned}
$$

3.6.1 Case 1

We take $a = 0$, $b = 0$, $\alpha = 0.2$. The measure we have for this example is:

$$\mu^* = 0.8821\delta_{-0.0314} + 0.1179\delta_{0.2321}$$

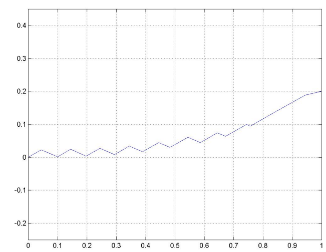

Figure 17: u(t) for the example 5, case 1

t	μ^*
0.1	$0.6808\delta_{-0.3727} + 0.3192\delta_{0.5003}$
0.2	$0.6706\delta_{-0.3729} + 0.3294\delta_{0.5002}$
0.3	$0.6648\delta_{-0.3732} + 0.3352\delta_{0.5001}$
0.4	$0.6632\delta_{-0.3734} + 0.3368\delta_{0.5003}$
0.5	$0.6660\delta_{-0.3738} + 0.3340\delta_{0.5004}$
0.6	$0.6729\delta_{-0.3742} + 0.3271\delta_{0.5002}$
0.7	$0.6841\delta_{-0.3746} + 0.3159\delta_{0.5003}$
0.8	$0.6997\delta_{-0.3750} + 0.3003\delta_{0.5009}$
0.9	$0.7197\delta_{-0.3755} + 0.2803\delta_{0.5009}$
1	$0.7443\delta_{-0.3759} + 0.2557\delta_{0.5000}$

Table 12: Table of measure

The figure 19 shows the oscillations for this problem.

3.6.2 Case 2

We take $a = 0.5$, $b = 0$, $\alpha = 0$. We obtain here the measure shown in the table 13. And the figure 20 we have the oscillations.

References

[Pe1997] Pedregal, P., Parametrized Measures and Variational Principles, Birkahauser, 1997.

[Pe2000] Pedregal, P., Variational Methods in Nonlinear Elasticity, SIAM, 2000.

[Da1999] Dacorogna, B., Direct Methods in the Calculus of Variations, Springer Verlag, 1989.

[Jo1998] Jost, J. and X. Li-Jost, Calculus of Variations, Cambridge studies in advanced mathematics No 64, Cambridge, 1998.

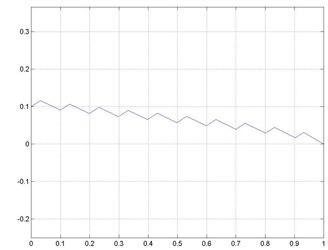

Figure 18: u(t) for the example 5, case 2

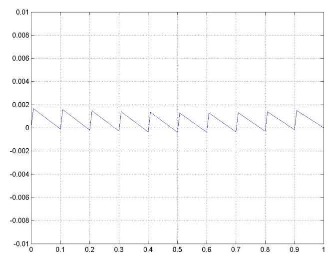

Figure 19: u(t) for the example 6, case 1

[Bu1998] Butazzo, G., M. Giaquinta and S. Hildebrandt, One-dimensional Variational Problems, Oxford Science Publications, Oxford University Press, 1998.

t	μ^*
0.1	$\delta_{-1.0151}$
0.2	$\delta_{-1.0250}$
0.3	$\delta_{-0.0000}$
0.4	$\delta_{-0.0000}$
0.5	$\delta_{-0.9237}$
0.6	$\delta_{-0.0000}$
0.7	$\delta_{0.0000}$
0.8	$\delta_{-0.0000}$
0.9	$\delta_{-1.6328}$
1	$\delta_{-0.3602}$

Table 13: Table of measure

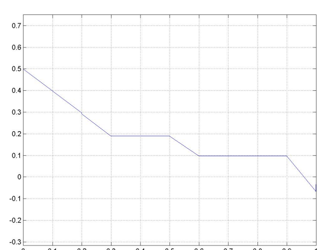

Figure 20: u(t) for the example 6, case 2