Control of Inventories under Non-Convex Polynomial Cost Functions

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Abstract

We propose an alternative method for computing effectively the solution of the control inventory problem under non-convex polynomial cost functions. We apply the method of moments in global optimization to transform the corresponding, non-convex dynamic programming problem into an equivalent optimal control problem with linear and convex structure. We device computational tools based on convex optimization, to solve the convex formulation of the original problem.

1 Introduction

This work proposes an alternative method for computing effectively the solution of the control inventory problem of the firm. The problem analyzed here is a version of the contrl inventory problem explored in [29], where the firm chooses inventories and production plannings to minimize the discounted present value of its costs. The essential difference between this and previous models is the presence of nonconvexities in the cost function of the technology facing firms; particular characteristics of internal labor markets as well as the capital utilization decisions of firms may alter the relationship between the level of output and costs, leading to non-convex k-time cost function, instead of the classical convex cost function with increasing marginal costs. Our aim in this work is to overcome this particular non convex situation from the point of view of optimization theory and convex analysis. We will focus on non convex expressions described by polynomials. We remark that non convex situations in optimization problems are difficult to understand. There is no general method for analyzing and solving problems with this feature [2, 4, 19, 20].

The economic literature has focused on the estimation of the effects of output, inventories and sales in the costs of the firm, using statistical and econometric techniques. However, in the case of the nonconvex cost function, it cannot be guaranteed that the calculated elasticities lie over the optimal solution, because sufficient conditions are not available [29]. To our knowledge, none of the existing works in the current economic research have focused on the computation of the level of inventories and production that minimize the costs of the firm, under non-convex costs of production. We accomplish this task by using the method of moments which allows us to find the optimal solution, in spite of the non convexities present in the dynamic programming model. This method has been recently proposed as a theoretical and practical tool for solving non convex optimization problems in control theory, global optimization and calculus of variations [3, 17, 18, 22–25, 27, 28, 32].

In this work we deal with non-convex dynamic programming problems given in the form:

$$\begin{aligned} &\min {I_N}^2 + \sum_{k=0}^{N} \beta^k C_k \\ \text{s.t.} \qquad & I_k - I_{k-1} = Y_k - S_k, \qquad k = 0, ..., N \qquad (1) \\ & I_{-1} = 0 \\ & I_k \geq 0, \quad Y_k \geq 0 \qquad\qquad k = 0, ..., N \qquad (2) \end{aligned}$$

where the cost function C_k can be expressed as a polynomial, namely

$$C_k = \gamma_3 Y_k^3 + \gamma_2 {Y_k}^2 + \gamma_1 (Y_k - Y_{k-1})^2 + \alpha_1 (I_k - \alpha_2 S_{k+1})^2 \qquad k = 0, ..., N$$

where Y_k is the production during the period k, I_k is the stock of finished goods inventories at the end of the period k, and S_{k+1} represents period $k+1$ sales; γ's and α's are theoretical parameters. We remark that the leading coefficient of the Hamiltonian function γ_3 is positive and γ_2 coefficient can be negative owed to negative marginal costos on the overrall economy.

The non-linear, non-convex form of the control variable, prevents us to use either the Hamiltonian equations of the minimum principle for discrete-time problems or mathematical programming techniques, because we cannot guarantee that the theoretical sufficient conditions for optimality hold on them. We propose to convexify the control variable Y_k by using the method of moments. By using the classical solution of the Stieltjes Moment Problem, we can formulate a linear, convex relaxation of (1) in the following form:

$$\begin{aligned} &\min {I_N}^2 + \sum_{k=0}^{N} \beta^k C''_k \\ \text{s.t.} \qquad & I_k - I_{k-1} = m_{1k} - S_k, \qquad k = 0, ..., N \\ & q_{k+1} = m_{1k}, \qquad k = 0, ..., N \qquad (3) \\ & I_{-1} = 0 \\ & I_k \geq 0, \quad m_{1k} \geq 0 \qquad\qquad k = 0, ..., N \qquad (4) \end{aligned}$$

where

$$\begin{aligned} C''_k = {} & \gamma_3 m_{3k} + (\gamma_1 + \gamma_2) m_{2k} - 2\gamma_1 q_k m_{1k} + \gamma_1 q_k^2 \\ & + \alpha_1 (I_k - a_2 S_{k+1})^2 \qquad k = 1, ..., N \end{aligned}$$

and the new control variable is the vector m_k in $\mathbb{R}^{L+1}$, whose i-entry is defined as

$$m_{ik} = \int_{\mathbb{R}_+} u_k^i d\mu(u_k) \qquad i = 0, ..., L.$$

That is, the entries of m_k are the moments of some measure μ with respect to the basis functions $\left\{1, u_k, u_k^2, ..., u_k^L\right\}$, supported on the semiaxis $[0, \infty)$ of the control variable Y_k.

What is new in this approach is the convexification of the control variable by using moment variables, which allows us to obtain an equivalent, convex formulation more appropriated to be solved by high performance numerical computing. We should warn the reader about the difficulties of numerical algorithms to overcome non-convex situations in optimization problems [9,14,19,21]. We will apply the Method of Moments to the control inventory problem under polynomial cost functions, using theoretical parameters.

The present paper is organized as follows. In section 2 we describe the control inventory problem explaining with some detail the theoretical source of the non-convexities in the cost function. In section 3 we outline the basics of the Method of Moments when the k-time objective function is a polynomial on the control variable. We also explain the essentials of the transformation of dynamic programming of problems like (1) into its equivalent linear, convex relaxation (3). In section 4 we motivate the application of the Method of Moments by calculating the convex envelope of a simpler k-time cost function and we solve the control inventory problem by using this method. We finish with a conclusion in Section 5.

2 The Control Inventory Problem

2.1 A Background on the Problem

The problem analyzed is a simple version of the control inventory problem explored in [29], where the firm chooses inventories and production to minimize the discounted present value of its costs. The difference between this and previous models is the nonconvexities in the technology facing firms. The standard production smoothing model of inventory investment states that a convex short-run cost function or a cost of changing the level of production induces firms to hold finished goods inventories in order to smooth production; this implies that production should not respond fully to a change in sales. However, certain overwhelming facts evidence that firms do not in fact smooth production; instead, production is actually more variable than sales and the covariance between sales and inventory change is not negative.

One possible explanation for the failure of the production smoothing model to explain the behavior of inventories is the presence of nonconvexities in the technology facing firms. The standard neoclassical theory states that it is optimal for firms to produce only over ranges of output where marginal costs are increasing, hence, over ranges where total costs are convex References [6,29,33] mention that the cost function may be convex at low output levels and concave at high levels. Then small shifts in demand could cause production to jump substantially. So it may be allowed to have declining marginal costs.

According to [29] characteristics of internal labor markets as well as the capital utilization decisions of firms alter the relationship between the level of output and costs. For example, the firm can treat high-skilled workers as fixed factors of production: during a downturn, it uses them for less productive tasks and when demand rises again, the firm can increase production by using the same labor inputs more efficiently; thus the enterprise is not forced to pay overtime work. Furthermore, processing firms may have incentives to build plants with large capacities, since the expanding of plants may increases production more than it rises costs.

Reference [29] is one of the leading papers taking into account non-convexities in the cost function of the inventory problem. In recent years several authors have been studying this kind of costs, and proposing different objective function forms. In [7] the authors study how plants in the U.S. automobile industry adjust production. They show that the lumpiness of the production in this industry is caused by exploitation of nonconvex operating margins; the main margins are adding or dropping a shift, varying regular hours by shutting the plant down for a week and, less important, overtime hours. These margins lead to nonconvexities in the cost function, which is the sum of several terms, one of them being non-convex and the rest being discontinuous in their arguments.

In [10] the authors investigate the aggregate implications of a nonconvexity in technology: the firm's choice of technique. In particular, they study a machine replacement problem in which a firm must decide whether or not to install a new machine or continue to produce with an older, depreciated machine. The empirical analysis focus on the U.S. automobile manufacturers and shows that the dramatic seasonal fluctuations of plants are induced by machine replacement, hence due to non-convexities in technology.

In [15] is presented econometric evidence of the incidence of non-convex costs in the relative variation of production to sales, in the automobile assembly plants. Following [7], the labor contract provisions and the non-convex margins produce large discontinuous jumps in the plant's cost curve. The author concludes that when desired production is above the plant's minimum efficient scale, non-convexities induce production bunching; the plant uses less than full capital utilization on average and production is more volatile than sales. When desired production is below the plant's minimum efficient scale, the plan operates in a convex region of the cost curve. In this case, it uses high levels of capital utilization and production is less volatile than sales.

In [30] the authors try to explain the decline in volatility of U.S. GDP growth beginning in 1984. In order to shed light into the discussion, they study the behavior of the U.S. automobile industry, where the changes in volatility have mirrored those of the aggregate data. They conclude that an inventory model involving non-convex costs predicts that a decline in the persistence of sales shocks leads to a decline in the variance of production relative to the variance of sales and to a decline in the covariance of inventory investment and sales.

From the preceding review it is clear that the recent literature has been looking for non-convexities in the U.S. automobile industry. According to [7] this is due to its substantial cyclical volatility and the quality of the data; but they note that the automobile industry is not representative to the entire economy and it is not the only industry where non-convex costs can be found. In fact, chapter 4.1 describes the evidence of non-convexities in the food industry presented in [29].

2.2 The Model

The model presented in [29] proposes a current-period cost function which takes the form of a three-degree polynomial on the control variable

$$C_k = \gamma_3 Y_k^3 + \gamma_2 {Y_k}^2 + \gamma_1 (Y_k - Y_{k-1})^2 + \alpha_1 (I_k - \alpha_2 S_{k+1})^2 \qquad k = 1, ..., N \quad (5)$$

where Y_k is the production during period k, I_k is the stock of finished goods inventories at the end of the period k, and S_{k+1} is period $k+1$ sales; γ's y α's are parameters.

The second term allows for the cost of producing Y_t, a convex cost in the short term; the third term represents the cost of changing the level of production (i.e. the cost of adjusting the labor force and reassigning tasks); the fourth term is a cost of deviating from target inventory, which is a linear function of sales; finally, the first term allows for a cubic cost function, which in the presence of the quadratic term (the cost of producing Y_t) with negative coefficiente may lead to a k–time non-convex cost function. Since the firm has not started the process of production at the beginning of the first period, the cost of changing the level of production at $k = 0$ is null. Hence, the current-period cost at $k = 0$ is

$$C_0 = \gamma_3 Y_0^3 + \gamma_2 {Y_0}^2 + \alpha_1 (I_0 - \alpha_2 S_1)^2$$

Given sales S_k, the firm chooses inventories and production to minimize the expected discounted present value of its costs, subject to the equation of inventory motion

$$\begin{aligned} & \min {I_N}^2 + \sum_{k=0}^{N} \beta^k C_k \\ \text{s.t.} \qquad & I_k - I_{k-1} = Y_k - S_k, \qquad k = 0, ..., N \\ & I_{-1} = 0 \\ & I_k \geq 0, \quad Y_k \geq 0 \qquad\qquad k = 0, ..., N \end{aligned} \quad (6)$$

where k is the time index, $0 < \beta < 1$ is a time discount factor and C_k is the current-period cost function during period k. The equation of inventory motion states that sales must be covered with output and inventories. We set the initial condition $I_{-1} = 0$ in order to impose a zero level of inventories at the beginning of the production process and we give to the the firm a punishment for holding high inventories at the end of its decision period - we assume this additional cost term is quadratic in the level of final inventories. Production Y_k is the control variable, inventories I_k represent the state variable and the sales S_k can be seen as an exogenous variable.

This problem is a simpler version of the one presented in [29]. The first difference lies in the omission of the price shock terms (relating wages, materials prices and energy prices) and the error terms; hence, our objective function is not the *expected* discounted value of costs but the *effective* discounted value of costs. The exclusion implies there is no uncertainty between agents. This assumption may be very restrictive, but its inclusion involves the analysis of stochastic optimization problems, which is beyond the scope of this paper. The second difference refers to the period length: we truncated the infinite period problem suggested in [29], in order to apply the minimum principle. The inclusion of the uncertainty terms and the infinite period treatment of the problem constitute items for future research.

3 Nonlinear Dynamic Programming Problems to the light of the Method of Moments

We develop here the discrete-time form of the Method of Moments, whose continuous version for optimal control problems is presented in [27]. The Method of Moments applies to dynamic programming problems given in the form:

$$\begin{aligned} \min J(u_0, ..., u_N) = f_{N+1}(x_{N+1}) + \sum_{k=0}^{N} f_k(x_k, u_k) \\ \text{s.t. } x_{k+1} = g_k(x_k, u_k), \qquad k = 0, ..., N \\ u_k \in U_k \subset \mathbb{R}_+, \qquad k = 0, .., N \\ x_0 = \bar{x}_0 \end{aligned} \tag{7}$$

where $(u_0, u_1, ..., u_N)$ is the control sequence, $(x_0, x_1, ..., x_{N+1})$ is the corresponding state sequence, the U_k are the control constraint sets which in this case all lie in the positive semiaxis on the real line[1] and the functions f_k and g_k can be expressed as polynomials in the control variable u_k; in general we have:

$$f_k(x_k, k, u_k) = \sum_{i=0}^{l_1} a_i(x_k, k) u_k^i \tag{8a}$$

$$g_k(x_k, k, u_k) = \sum_{i=0}^{l_2} c_i(x_k, k) u_k^i. \tag{8b}$$

[1] The Method of Moments deals with constraint sets lying not only in the semiaxis $[0, \infty)$, but in the entire real space. Given the characteristics of the control inventory problem, however, it is necessary to study this more restrictive problem.

We assume that f_k and g_k functions are continuously differentiable with respect to x_k. The nonconvex form of the control variable is an obstacle to use the Hamiltonian function of the minimum principle[2] and non-linear mathematical programming techniques. An alternative approach for dealing with this kind of problems is to convexify the control variable by using the method of moments in the polynomial expressions (8).

Given the polynomial form of f_k and g_k, the Hamiltonian H_k of the dynamic programming problem is a polynomial in the control variable:

$$H_k = \sum_{i=0}^{L} \alpha_i(x_k, p_k, k) u_k^i \qquad k = 0, ..., N \tag{9}$$

where $L = \max\{l_1, l_2\}$. We are interested here in finding the global minimization of H_k in u_k, namely

$$\min_{u_k \geq 0} H_k(u_k) = \sum_{i=0}^{L} \alpha_i u_k^i. \tag{10}$$

Notice that we use positive controls as this is the common setting in inventory models.

3.1 The General Theory of the Method of Moments

One approach for solving problem (10) comes from convex analysis, because we can use the convex envelope of the function H_k in order to locate its global minima [23]. The following theorem characterizes the convex envelope of the Hamiltonian function by using measure theory.

Theorem 1 *The convex envelope of $Epi(H_k)$ can be expressed as*

$$co(Epi(H_k)) = \left\{ \int_{\mathbb{R}_+} H_k(u_k) d\mu(u_k) : \mu \in P(\mathbb{R}_+) \right\}$$

where $P(\mathbb{R}_+)$ is the family of all probability Borel measures supported in the semiaxis $[0, \infty)$.

Once we have characterized the convex hull of H_k, we can obtain the set of all global minima of H_k using a recent result for global optimization of polynomials.

Theorem 2 *[17, 22] Let $P(\mathbb{R}_+)$ be the set of all regular Borel probability measures supported in the semiaxis $[0, \infty)$. If H_k is an algebraic polynomial whose leader coefficient α_L is positive , then*

$$\min_{\mu \in P(\mathbb{R}_+)} \int_{\mathbb{R}_+} H_k(u_k) d\mu(u_k) = \min_{u_k \in \mathbb{R}_+} H_k(u_k).$$

[2] For a description of the minimum principle for discrete-time problems see [5] and [8].

From the previous theorem, it follows that we should use the generalized optimization problem in measures

$$\min_{\mu \in P(\mathbb{R}_+)} \int_{\mathbb{R}_+} H_k(u_k) d\mu(u_k) \tag{11}$$

as an alternative formulation of the global optimization problem (10). The following theorem states that the solution of problem (11) is the family of all probability measures supported in the set of all global minima of the Hamiltonian function H_k.

Theorem 3 *[17,23,24] Let G be the set of all global minima of the Hamiltonian function H_k in $\mathbb{R}_+$ then,*

$$\int_{\mathbb{R}_+} H_k(u_k) d\mu^*(u_k) = \min_{\mu \in P(\mathbb{R}_+)} \int_{\mathbb{R}_+} H_k(u_k) d\mu(u_k)$$

if and only if the support of μ^* is contained in G. Briefly, the set $P(G)$ is the solution set for the generalized problem (11).

The preceding two theorems show that there exists a theoretical equivalence between the minimization problem (10) and the relaxed problem (11). The following section make such equivalence explicit and useful.

3.2 Convexification of Polynomial Expressions

The relaxed problem (11) contains information about all the global minima of the function H_k in $\mathbb{R}_+$. This kind of problems cannot be solved easily in practice, due to the difficulty for describing all possible convex combinations of points in $\mathbb{R}_+$. However, the polynomial form of the Hamiltonian function $H_k(u_k) = \sum_{i=0}^{L} \alpha_i u_k^i$ make it more manageable.

Every integral in problem (11) can be expressed as an elementary dot product in $\mathbb{R}^{L+1}$

$$\int_{\mathbb{R}_+} H_k(u_k) d\mu(u_k) = \sum_{i=0}^{L} \alpha_i m_{ik} = \alpha \cdot m_k$$

where α is the coefficients vector of the Hamiltonian polynomial at time k and the moment vector m_k in $\mathbb{R}^{L+1}$ is defined as:

$$m_{ik} = \int_{\mathbb{R}_+} u_k^i d\mu(u_k) \qquad i = 0, ..., L \tag{12}$$

which are the moments of some measure μ with respect to the functional basis

$$\{1, u_k, u_k^2, ..., u_k^L\}$$

at time k.

The optimization problem (11) transforms into the following convex problem:

$$\min_{m_k \in M} \sum_{i=0}^{L} \alpha_i m_{ik} \tag{13}$$

where M is the convex cone of all vectors in $\mathbb{R}^{L+1}$ whose entries are the algebraic moments of a probability measure supported in $\mathbb{R}_+$, namely

$$M = \{m \in \mathbb{R}^{L+1} : m_i = \int_{\mathbb{R}_+} u^i d\mu(u) \quad i = 0, ..., L \ , \ \ \mu \in P(\mathbb{R}_+)\}.$$

Although this formulation seems attractive due to the linear form of the objective function and the convex structure of the feasible set, it is still a theoretical formulation not very useful if we do not properly characterize the feasible set M, see [27]. The characterization of the values $m_{0k}, m_{1k}, ..., m_{Lk}$, as the moments of some measure μ, is an open question in contemporary mathematics. This difficult task is called *The Problem of Moments*. Given the standard algebraic basis in $\mathbb{R}_+$ and values $m_{0k}, m_{1k}, ..., m_{Lk}$, the Problem of Moments consists in determining a positive measure μ such that equation (12) holds; it also includes the search for requirements in order to characterize $m_{0k}, m_{1k}, ..., m_{Lk}$ as a set of moments. Depending on the functional basis and the domain set the Problem of Moments can take different forms. For the standard algebraic basis and the domain $\mathbb{R}_+$ the Problem of Moments is referred as *Stieltjes Moment Problem.*

The solution of the Stieltjes Moment problem is summarized in the following result:

Lemma 4 *[16] Let $L = 2n + 1$ and consider the matrices $A_k = (m_{i+j,k})_{i,j=0}^n$ and $B_k = (m_{i+j+1,k})_{i,j=0}^n$. If matrices A_k and B_k are both positive definite then the vector $(m_{0k}, ..., m_{Lk})$ is in $\bar{M}$. Conversely, if $m_{0k}, ..., m_{Lk}$ are the algebraic moments of some positive measure supported in $[0, \infty)$, then the matrices A_k and B_k are positive semidefinite.*

From the preceding lemma we conclude that the closure of M is composed of all vectors in $\mathbb{R}^{L+1}$ whose entries form two positive semidefinite (p.s.d) Hankel matrices [27]:

$$\bar{M} = \{(m_i)_{i=0}^{2n+1} \in \mathbb{R}^{L+1} : (m_{i+j})_{i,j=0}^n, (m_{i+j+1})_{i,j=0}^n \text{ are p.s.d with } m_0 = 1\}.$$

This result allows us to transform the relaxed problem (13) into the semidefinite program:

$$\min_{m_k} \sum_{i=0}^{2n+1} \alpha_i m_{ik} \tag{14}$$

$$\text{s.t.} \quad (m_{i+j,k})_{i,j=0}^n \geq 0, (m_{i+j+1,k})_{i,j=0}^n \geq 0, \text{ with } m_0 = 1.$$

So far, we have presented the existing links between the non-linear, non-convex problem (10) and the convex program (14). Now we want to guarantee we can obtain some information about the global minima of the later by solving the former. This task is accomplished in [23].

Theorem 5 *[23] Let μ_k^* be one probability measure supported in $[0,\infty)$, whose supporting points are global minima of the Hamiltonian function $H_k(u_k)$ in $[0,\infty)$. Then, the algebraic moments $m_{0k}^*, ..., m_{Lk}^*$ of the measure μ_k^* solve the optimization problem (14). At the converse, if values $m_{0k}^*, ..., m_{Lk}^*$ solve problem (14), there exists a unique probability measure μ_k^* supported in $[0,\infty)$, with algebraic moments $m_{0k}^*, ..., m_{Lk}^*$, whose supporting points are global minima for $H_k(u_k)$ in $[0,\infty)$.*

Moreover, we can relate the minimizers of problem (14) with the number of global minima present in the objective function of the problem (10), as the following corollaries states.

Corollary 6 *[27] Since the set of global minima of H_k is finite, any solution m_k^* of problem (14) can be expressed as*

$$m_k^* = \lambda_1 T(u_{1k}) + ... + \lambda_s T(u_{sk})$$

where $u_{1k}, ..., u_{sk}$ are global minima of H_k, $s \leq L$, $\lambda_j > 0$ with $\sum_{j=1}^{s} \lambda_j = 1$, where T is the nonlinear transformation $T : \mathbb{R}_+ \to \mathbb{R}^{L+1}$ defined by the expression $T(u_k) = (1, u_k^2, ..., u_k^L)$.

Therefore, if H_k has a unique global minimum u_k^*, the optimal control can be expressed as $u_k^* = m_{1k}^*$. According to the preceding corollary, we can also state that any solution μ_k^* of problem (11) can be expressed as $\mu_k^* = \lambda_1 \delta_{u_{1k}^*} + ... + \lambda_s \delta_{u_{sk}^*}$. When u_k^* is the unique global minimum we have $\mu_k^* = \delta_{u_k^*}$, where δ_t represent a dirac measure.

3.3 Analysis of the problem

In the previous section we outlined the basics of the Method of Moments. These results suggest to reformulate the global minimization of the Hamiltonian H_k in problem (10) as:

$$\min_{m_k} H_k(x_k, k, p_k, m_k) = \sum_{i=0}^{2n+1} \alpha_i(x_k, k, p_k) m_{ik} \tag{15}$$

$$\text{s.t.} \quad (m_{i+j,k})_{i,j=0}^{n} \geq 0, (m_{i+j+1,k})_{i,j=0}^{n} \geq 0, \text{ with } m_{0k} = 1 \qquad k = 0, ..., N.$$

Then, we can solve the non-liner, non-convex problem (7) by dealing with its convex relaxation:

$$\begin{aligned} &\min_{m_k} \; f_{N+1}(x_{N+1}) + \sum_{k=0}^{N} \sum_{i=0}^{l_1} a_i(x_k, k) m_{ik} \\ \text{s.t.} \quad & x_{k+1} = \sum_{i=0}^{l_2} c_i(x_k, k) m_{ik} \\ & (m_{i+j,k})_{i,j=0}^{n} \geq 0, (m_{i+j+1,k})_{i,j=0}^{n} \geq 0, \text{ with } m_{0k} = 1 \\ & \text{for } \; k = 0, ..., N \\ & x_0 = \bar{x}_0 \end{aligned} \tag{16}$$

where $L = \max\{l_1, l_2\} = 2n + 1$.

Notice that the semidefinite program (15) corresponds to the optimization of the Hamiltonian of the convex formulation (16)

$$\begin{aligned}\widetilde{H}_k &= \widetilde{H}_k(x_k, k, p_k, m_k) = \sum_{i=0}^{l_1} a_i(x_k, k) m_{ik} + p_k^{'} \cdot \sum_{i=0}^{l_2} c_i(x_k, k) m_{ik} \\ &= \sum_{i=0}^{L} \alpha_i(x_k, k, p_k) m_{ik}\end{aligned}$$

The analytical aspects of the formulation (16) and its relation with problem (7) are a consequence of the results presented in the previous section. The following theorem and its corollary provide a practical method to certify if problem (7) lacks of minimizers.

Theorem 7 *Let us assume that u_k^* is a minimizer of the optimal control problem (7), then the control vector m_k^* given as*

$$m_{ik}^* = (u_k^*)^i \qquad \textit{for } i = 0, ..., L. \tag{17}$$

is a minimizer of the formulation (16).

Proof. This proof follows the one outlined in [27] for optimal control problems. Since u_k^* is an optimal control for problem (7), according to [8] the minimum principle claims that u_k^* satisfies the global minimization problem:

$$H_k(x_k^*, k, p_k^*, u_k^*) = \min_{u_k \in \mathbb{R}^+} H_k(x_k^*, k, p_k^*, u_k) \tag{18}$$

where x_k^* comes from the solution of the differential equation:

$$\begin{gathered}x_{k+1}^* = g_k(x_k^*, u_k^*) \qquad \text{for} \quad k = 0, ..., N \\ x_0^* = \bar{x}_0\end{gathered}$$

and the function p_k^* comes from the solution of the finite differences equation:

$$p_k^* = \frac{\partial H_{k+1}(x_{k+1}^*, u_{k+1}^*, p_{k+1}^*, k+1)}{\partial x_{k+1}} \qquad \text{for} \quad k = 0, ..., N-1$$

and the boundary condition

$$p_N^* = \frac{df_{N+1}(x_{N+1}^*)}{dx_{N+1}}.$$

On the other hand, the Hamiltonian function H_k has the polynomial form (9) and u_k^* solves the global minimization problem (18), therefore we can use the theory of global optimization of polynomials of Section 3 to show that the vector $m_k^* \in \mathbb{R}^{L+1}$ defined as:

$$m_{ik}^* = (u_k^*)^i \qquad i = 0, ..., L$$

satisfies the semidefinite program:

$$\min_{m_k \in \bar{M}} \widetilde{H}_k(x_k^*, k, p_k^*, m_k) \tag{19}$$

where:

$$\widetilde{H}_k(x_k, k, p_k, m_k) = \sum_{i=0}^{L} \alpha_i(x_k, k, p_k) m_{ik}$$

and the functions x_k^*, p_k^* in problem (19) come from the solution of the finite differences equations:

$$\begin{aligned} x_{k+1}^* &= g_k(x_k^*, u_k^*) \qquad \text{for} \quad k = 0, ..., N \\ p_k^* &= \frac{\partial H_{k+1}(x_{k+1}^*, u_{k+1}^*, p_{k+1}^*, k+1)}{\partial x_{k+1}} \qquad \text{for} \quad k = 0, ..., N-1 \end{aligned} \tag{20}$$

with the boundary conditions $x_0^* = \bar{x}_0$ and $p_N^* = \frac{df_{N+1}(x_{N+1}^*)}{dx_{N+1}}$. Since g_k and $\frac{\partial H_k}{\partial x_k}$ in system (20) have polynomial form in the variable u_k^*, and every appearance of the *i-th* power of u_k^* can be replaced by m_{ik}^*, then we can see that functions x_k^* and p_k^* satisfy the differential equations:

$$\begin{aligned} x_{k+1}^* &= \widetilde{g}_k(x_k^*, m_k^*) \qquad \text{for} \quad k = 0, ..., N \\ p_k^* &= \frac{\partial \widetilde{H}_{k+1}(x_{k+1}^*, m_{k+1}^*, p_{k+1}^*, k+1)}{\partial x_{k+1}} \qquad \text{for} \quad k = 0, ..., N-1 \end{aligned}$$

where $\widetilde{g}_k(x_k, m_k) = \sum_{i=0}^{l_2} c_i(x_k, k) m_{ik}$ and $\frac{\partial \widetilde{H}_k}{\partial x_k}$ is the formal partial derivative of $\widetilde{H}_k$ with respect to variable x_k. Since m_k^* solves the program (19), we have:

$$H_k(x_k^*, k, p_k^*, m_k^*) = \min_{m_k \in \bar{M}} H_k(x_k^*, k, p_k^*, m_k)$$

and we conclude that m_k^* satisfies the minimum principle's necessary conditions for the minimizers of the convex formulation (16). Thus, m_k^* must be a minimizer of (16). ■

Note that if m_k^* is a minimizer of problem (16) satisfying (17), then $(m_{1k}^*)^i = m_{ik}^*$ and m_{1k}^* is a minimizer of problem (7). This situation is particularly convenient in order to calculate minimizers of problem (7), because we only have to solve the convex formulation (16) which is more appropriated to be handled by high performance, non-linear programming techniques [27].

Corollary 8 *[27] If all the minimizers of the formulation (16) fail in satisfying the expression (17), the problem (7) lacks of minimizers.*

Therefore we can determine the lack or existence of minimizers in problem (7) by checking all the minimizers of formulation (16) satisfying $(m_{1k}^*)^i = m_{ik}^*$ $i = 1, ..., L$.

4 Solving the Control Inventory Problem with the Method of Moments

In this section we solve the control inventory problem (6) by using the Method of Moments. In order to justify the application of this method, we will describe where do the non-convexities arise and we mention its effects on the decision problem of the firm. Then, we calculate the optimal production and inventories for each period k.

First we transform slightly problem (6). Note that the k-time cost function (5) depends not only on the current control Y_k but also on the lagged control Y_{k-1}; however, the minimum principle admits only current values of controls and states. Following [5] this situation can be handled by introducing a new state variable. Thus, we introduce $q_k = Y_{k-1}$ and add the equation $q_{k+1} = Y_k$ to the system of inventory motion. After some algebraic manipulation, the optimization problem (6) yields,

$$
\begin{aligned}
& \min {I_N}^2 + \sum_{k=0}^{N} \beta^k C'_k \\
\text{s.t.} \qquad & I_k - I_{k-1} = Y_k - S_k, \qquad k = 0, ..., N \\
& q_{k+1} = Y_k, \qquad k = 0, ..., N \\
& I_{-1} = 0 \\
& I_k \geq 0, \quad Y_k \geq 0 \qquad\qquad k = 0, ..., N
\end{aligned}
\tag{21}
$$

where

$$
\begin{aligned}
C'_k &= \gamma_3 Y_k^3 + (\gamma_1 + \gamma_2) Y_k^2 - 2\gamma_1 q_k Y_k + \gamma_1 q_k^2 \\
&\quad + \alpha_1 (I_k - a_2 S_{k+1})^2 \qquad k = 1, ..., N \\
C'_0 &= \gamma_3 Y_0^3 + \gamma_2 Y_0^2 + \alpha_1 (I_0 - \alpha_2 S_1)^2
\end{aligned}
\tag{22}
$$

Note that the constraint sets for each control variable Y_k are simply $\mathbb{R}^+$ for $k = 0, .., N$ [3]; they are indeed convex as is required by the discrete-time minimum principle [5]. Also note that the cost function (22) is not convex on the control variable Y_k. Since we cannot use the sufficient conditions of the minimum principle, we cannot apply it to problem (21) as we are not certain about the optimality condition.

4.1 The non-convex cost function

Now we outline some theoretical and empirical facts in order to understand where do the non-convexities of the cost function arise. Reference [29] states that in the presence of certain technologies it is possible to have both declining and increasing marginal costs over some ranges of production. A cubic term on the production Y_k was included in the k-time cost function (22) -besides

[3] This is due to the natural assumption of positive production.

the quadratic term- which can entail nonconvexities in the cost function. A negative $\gamma_1 + \gamma_2$, which is the coefficient of the quadratic term on the cost function (22), would evidence the presence of declining marginal costs; if in addition the coefficient of the cubic term γ_3 is positive, we can state that the cost function is nonconvex in the positive real line.

To illustrate this situation, figure 1 presents the k-time cost function (22) including only the cubic and quadratic terms[4], with a positive coefficient for the former $\gamma_3 = 0.1$, and a negative coefficient for the later $\gamma_1 + \gamma_2 = -6$. The sum of the convex function $F(Y) = 0.1Y^3$ and the concave function $G(Y) = -6Y^2$ is the non-convex function $C(Y) = 0.1Y^3 - 6Y^2$.

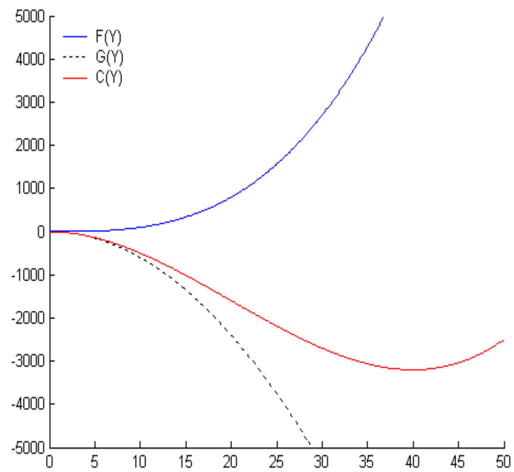

Figure 1: $F(Y) = 0.1Y^3$, $G(Y) = -6Y^2$, $C(Y) = 0.1Y^3 - 6Y^2$

Table 1 presents all the parameters of the cost function calculated in [29] using statistical and econometric techniques; the industries studied in [29] are food, tobacco, apparel, chemicals, petroleum, rubber and automobile. In all seven industries $\gamma_1 + \gamma_2$ was estimated to be negative, indicating declining marginal costs. In all industries but food, γ_3 was not significantly different from zero. We remark these estimations imply that the non-convexities in the cost function arise only in the food industry. The other six industries present concave cost functions, where the cost minimization problem makes no sense; because its leader coefficient $\gamma_1 + \gamma_2$ is negative, the k-time polynomial cost function is unbounded from below[5].

[4] The omitted terms would translate the curve or change slightly its slope, but the concavity of the function would not be changed.

[5] Note that this results contradict the standard neoclassical theory, which states that the firm always faces increasing marginal costs.

	$\boldsymbol{\gamma}_1$	$\boldsymbol{\gamma}_2$	$\boldsymbol{\gamma}_3$	$\boldsymbol{\alpha}_1$	$\boldsymbol{\alpha}_2$
Food	1.100000	-7.780000	0.000307	1.000000	0.857000
Tobacco	0.133000	-0.740000	0.000000	1.000000	0.407000
Apparel	0.094800	-0.531000	0.000000	1.000000	0.314000
Chemical	0.014400	-0.566000	0.000000	1.000000	0.389000
Petroleum	0.075800	-0.354000	0.000000	1.000000	-0.040000
Rubber	-0.009500	-0.218000	0.000000	1.000000	0.436000
Autos	0.477000	-1.780000	0.000000	1.000000	0.415000
Theoretical	1.000000	-7.000000	0.100000	1.000000	1.000000

Table 1: Parameters of the Cost Function

Then, we should only study those minimization problems where the parameters of the polynomial cost function exhibit non-convex (non-concave) behavior through their cost function. For the food industry, the difference in the order of magnitude between γ_3 and the other coefficients makes it difficult to calculate the solution of the optimization problem with the available computational tools -see Table 1. Thus, in order to make the problem manageable, we use theoretical parameters for the polynomial cost function, presented in Table 1, as a qualitative model of this situation. Note that the selected parameters are quite similar from those of the food industry, except for γ_3 which is higher to avoid problems with its order of magnitude.

4.2 The Convex Envelope of the Cost Function

In this section we describe the effects of the non-convexities of the cost function in the decision problem of the firm and we outline the role of its convex envelope when calculating the optimal solution. In the previous subsection we explained how this non-convexities arise. The analysis of the effects of the non-convexities in this simpler case, may help us to understand better its effects in the more complicated inventory control problem.

Suppose the firm faces a non-convex cost function given by

$$C = 0.1Y^3 - 6Y^2 \tag{23}$$

like the one illustrated in figure 1. In order to minimize its costs, it is not optimal for the firm to choose those ranges of production where the curve is non-convex; instead, it should choose only those ranges of output where the cost function is convex. That is, in the minimization process the firm should use not the non-convex cost function, but its convex envelope.

4.2.1 Analytical Frame for the Convex Envelope

Given any function $f : [0,\infty) \to \mathbb{R}$, its convex envelope is characterized as a convex function $f_c : [0,\infty) \to \mathbb{R}$ which makes true the following expression: $Epi(f_c) = \overline{co(Epi(f))}$. By using Caratheodory's theorem of convex analysis, we conclude that every point $(a, f_c(a))$ on the graph of the convex envelope of f, can be expressed as a convex combination of points located on the graph of f with two terms at most. Then, we can express $(a, f_c(a))$ as

$$(a, f_c(a)) = \lambda_1(a_1, f(a_1)) + \lambda_2(a_2, f(a_2)) \tag{24}$$

where $\lambda_1 + \lambda_2 = 1$, $\lambda_1, \lambda_2 \geq 0$ and $a_i \in \mathbb{R}^+$ for $i = 1, 2$. Following [26], if we interpret the points a_1, a_2 and the coefficients λ_1, λ_2 as the components of a discrete probability distribution, that is

$$\mu^* = \lambda_1 \delta_{a_1} + p_2 \delta_{a_2} \tag{25}$$

where δ_a represents a Dirac measure, then we can express (24) as a particular integration process:

$$(a, f_c(a)) = \int_0^\infty (t, f(t)) d\mu^*(t)$$

where μ^* is supported in $[0, \infty)$ -because $a_1, a_2 \in [0, \infty)$.

We can estimate the values a_1, a_2 and λ_1, λ_2 of (24), by solving the following optimization problem:

$$f_c(a) = \min_\mu \int_0^\infty f(t) d\mu(t) \tag{26}$$

where μ represents the family of probability distributions in $[0, \infty)$ with mean a [31]. If we use the fact that f can be described as a polynomial of odd degree given in the general form:

$$f(t) = \sum_{i=0}^{2n+1} c_i t^i$$

then we can transform the optimization problem (26) into the optimization problem

$$f_c(a) = \min_m \sum_{i=0}^{2n+1} c_i m_i \tag{27}$$

where the variables m must belong to the convex set M of all the vector $m \in \mathbb{R}^{2n+2}$ whose entries are the first $2n + 2$ algebraic moments of one positive measure supported in $[0, \infty)$ with mean equal to a [1, 11, 16]. By using the solution of the *Stieltjes Moment Problem* -see section 3.2- the optimization problem (27) is equivalent to the following semidefinite program:

$$\begin{gathered}
\min_m \sum_{i=0}^{2n+1} c_i m_i \\
\text{s.t.} \quad (m_{i+j})_{i,j=0}^n \geq 0, \quad (m_{i+j+1})_{i,j=0}^n \geq 0 \\
\text{with} \quad m_0 = 1 \text{ and } m_1 = a.
\end{gathered}$$

Since μ^* is supported in two points at most, we can construct μ^* in (25) by using its moments $1, a, m_2^*, m_3^*$ obtained after solving the semidefinite program (27). This task can be carried out by elementary algebra [1, 11, 16].

4.2.2 The Simpler Cost Function and its Convex Envelope

By using the method described previously, we graph the convex envelope C_c of the simpler cost function (23) in Figure 2. For a given point in the domain, for example $a = 15,$ we obtain the coefficients $\lambda_1 = 0.4966$, $\lambda_2 = 0.5034$ and the points $a_1 = 0.0219$, $a_2 = 29.7785$. That is, the point $(15, C_c(15))$ in the convex envelope of C can be expressed as the convex combination of two points located in the graph of C

$$(15, C_c(15)) = 0.4966 \cdot (0.0219, C(0.0219)) + 0.5034 \cdot (29.7785, C(29.7785))$$

whose associated discrete probability distribution is

$$\mu^* = 0.4966 \cdot \delta_{0.0219} + 0.5034 \cdot \delta_{29.7785}$$

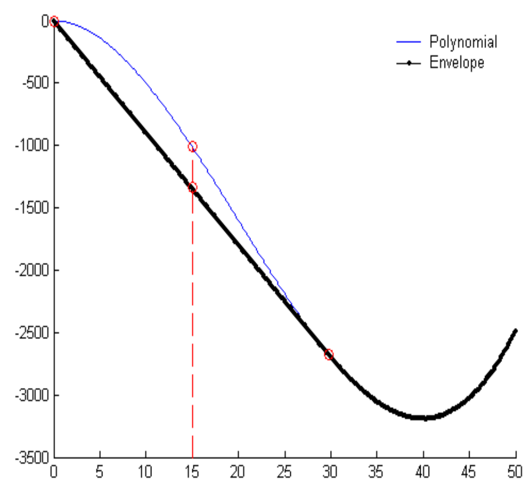

Figure 2: $C = 0.1Y^3 - 6Y^2$ and its convex envelope C_c

As we noted earlier, it is not optimal for the firm to choose levels of production were the convex envelope C_c differs from the cost function (23). In this simpler static problem, it is clear which levels of output should the firm choose: those which minimize the convex envelope of the cost function. The same notion applies for our original optimization problem (21), however, it will not be that clear which level of production must the firm choose, due to the dynamic nature of our problem. The optimal solution can be obtained by using the Method of Moments described in section 3, which minimizes the convex envelope of the k-time cost function, but in a dynamic frame.

4.3 The Solution of the Control Inventory Problem

Now we solve the control inventory problem (21) by using the Method of Moments. Since the control variable must lay in $\mathbb{R}^+$, all moment vectors of positive measures are supported in the semiaxis $[0,\infty)$, then we use the solution of the odd case of the Stieltjes Problem[6]. Doing so, the convex relaxation of problem (21) yields

$$
\begin{aligned}
&\min_{m_k} {I_N}^2 + \sum_{k=0}^{N} \beta^k C_k'' \\
\text{s.t.}\quad & I_k - I_{k-1} = m_{1k} - S_k, \qquad k = 0,...,N \\
& q_{k+1} = m_{1k}, \qquad k = 0,...,N \\
& I_{-1} = 0 \\
& I_k \geq 0, \quad m_{1k} \geq 0 \qquad\qquad k = 0,...,N \\
& (m_{i+j,k})_{i,j=0}^{1} \geq 0, (m_{i+j+1,k})_{i,j=0}^{1} \geq 0,\ m_{0k} = 1 \qquad k = 0,...,N
\end{aligned}
\tag{28}
$$

where

$$
\begin{aligned}
C_k'' &= \gamma_3 m_{3k} + (\gamma_1 + \gamma_2) m_{2k} - 2\gamma_1 q_k m_{1k} + \gamma_1 q_k^2 \\
&\quad + \alpha_1 (I_k - a_2 S_{k+1})^2 \qquad k = 1,...,N \\
C_0'' &= \gamma_3 m_{30} + \gamma_2 m_{20} + \alpha_1 (I_0 - \alpha_2 S_1)^2
\end{aligned}
\tag{29}
$$

The optimization problem (28) is a non-linear mathematical program. Following [27], in order to represent the matrix inequality conditions as a set of non-linear inequalities, we use the fact that all subdeterminants of a positive semidefinite matrix are nonnegative [12]. Then, the matrix inequality conditions $(m_{i+j,k})_{i,j=0}^{1} \geq 0, (m_{i+j+1,k})_{i,j=0}^{1} \geq 0$ given as

$$
\begin{bmatrix} m_{0k} & m_{1k} \\ m_{1k} & m_{2k} \end{bmatrix} \geq 0, \begin{bmatrix} m_{1k} & m_{2k} \\ m_{2k} & m_{3k} \end{bmatrix} \geq 0
$$

are expressed as a set of non-linear inequality constraints:

$$
\begin{array}{cc}
m_{0k} \geq 0 & m_{1k} \geq 0 \\
m_{2k} \geq 0 & m_{3k} \geq 0 \\
m_{0k} m_{2k} - m_{1k}^2 \geq 0 & m_{1k} m_{3k} - m_{2k}^2 \geq 0
\end{array}
$$

Hence, we have transformed the optimal control problem (6) into a non-linear, convex, mathematical program in $5 \times (N+1)$ variables and $9 \times (N+1)$ constraints. Notice that the independent coefficient of the cost function depends on I_k and q_k. In order to solve this kind of high dimensional, non-linear mathematical programs, we use standard professional software based on Sequential Quadratic Programming [9, 14, 19, 21].

[6] Note that the assumption of convext constraint sets is not an obstacle for applying the Method of Moments in this particular problem.

The parameters of the polynomial cost function are presented in Table 1, which are quite similar from those of the food industry, as noted earlier. Following [29], the discount factor β was preset at 0.99. We choose monthly sales, which is the exogenous variable, in four different scenarios. The first scenario assumes stable sales; the second scenario, ascending sales; the third, descending sales; and the fourth, variable sales. Their order of magnitude was chosen according to the level of production which minimizes the simpler cost function (23) -according to Figure 2, this optimal output is approximately 40.

We checked the uniqueness of the minimizers by evaluating expression (17); when the order of magnitude of expressions

$$(m_{1k})^2 - m_{2k}, \quad (m_{1k})^3 - m_{3k} \qquad k = 0, ..., N \tag{30}$$

was small we concluded that the minimizers were unique.

We solved the optimization problem along 12 months. We tried to solve the problem for 24 months, but we found that in this case the problem lacks of minimizers in the four scenarios. A possible explanation for this finding is that production and inventories decisions are only made in the short run, because firms cannot forecast sales with precision beyond 12 months.

We remark that it is possible to calculate generalized solutions when the problem lacks of minimizers; that is, the solution of problem (11) may be described as a convex combination of Dirac measures supported in the global minima of the cost function. However, this issue is out of the scope of this paper and it can be accomplished in future research. For future references see [23, 24, 27].

4.3.1 Optimal Solution with Stable Sales

We constructed the vector of stable sales using observed shipments growth between 1998 and 2000 for the food industry in the United States. We obtained unique optimal solution when sales are stable; the order of magnitude of expression (30) is small -see Appendix A. Figure 3 presents optimal production and inventories. The optimal decision consists in producing more than the demanded output in the first periods, resulting in an accumulation of inventories. In the subsequent periods, the firm must decrease its production below demanded output; then, it must satisfy sales with accumulated inventories. The stock of inventories diminishes on the last months in order to avoid the higher costs this holding implies.

4.3.2 Optimal Solution with Ascending Sales

When sales increase, we obtained unique optimal solutions; the order of magnitude of expression (30) is small - see Appendix A. Figure 4 presents optimal production and inventories for this scenario. The former increases with sales and its level is above demand in the first periods, resulting in an accumulation of inventories. After month 10, the firm must decrease its output, satisfying sales with its holding of inventories. We remark that the accumulation of inventories is higher than in the previous scenario, in order to satisfy ascending

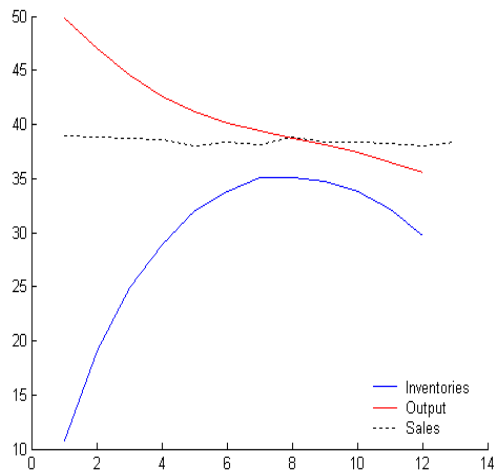

Figure 3: Optimal solution with stable sales

sales; however, as in the stable sales scenario, the firm decreases its holding of inventories in the final periods in order to avoid the higher costs it implies.

4.3.3 Optimal Solution with Descending Sales

When sales decrease, we found no minimizers; the order of magnitude of expression (30) is huge -see Appendix A. A possible explanation for this finding is that when demand diminishes with a constant rate along 12 months, it is hard for the firm to avoid the costs of decreasing sharply production or increasing rapidly its holding of inventories. These results seems coherent, since it may be unsustainable for any firm to face persistent decreasing sales.

4.3.4 Optimal Solution with Variable Sales

We found unique minimizers when sales are variable -see Appendix A. Figure 5 presents optimal production and inventories. In this case, the higher variance of the demand is absorbed by inventories, which vary more than optimal output. This supports the production smoothing model, which states that when firms hold inventories, production may not respond fully to changes in sales. As we mentioned in section 2, the introduction of non-convexities in the cost function where motivated by the failure of this model; that is, the empirical facts evidence that firms do not smooth production, contradicting our findings. This may be due to the huge variance of our theoretical sales, which is higher than the variance of real sales; in fact, the rate of growth of real sales were used to construct the stable sales vector in the first scenario. This finding constitutes

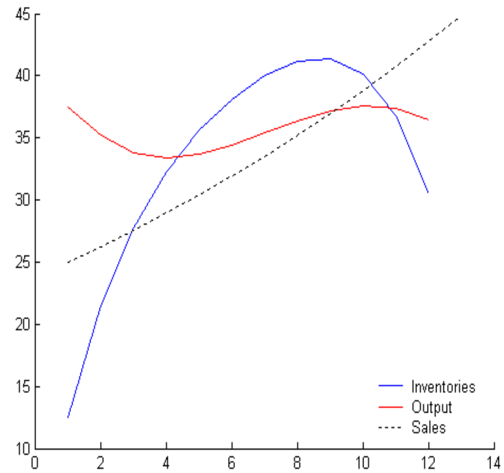

Figure 4: Optimal solution with ascending sales

a contribution to the production smoothing model: it is fully functional only when sales are highly variable.

The optimal decision consists in producing more when sales are low, allowing the accumulation of inventories. When demand rises sharply, the firm satisfies it not by increasing production but by using its accumulated inventories. In the last month the firm must decrease its holding of inventories in order to avoid higher costs.

5 Concluding Remarks

In this work we have proposed a new method for solving explicitly the control inventory problem, where the firm chooses the level of production and inventories which minimizes the discounted present value of its costs. Our problem is a simple version of the one studied in [29], where the instantaneous cost function is a non-convex odd-degree polynomial in the control variable (production). Since the k-time objective function is non-convex, the minimum principle for discrete-time problems cannot provide sufficient conditions for optimality. Hence, following [27], we apply the Method of Moments to our problem and provide necessary and sufficient conditions for the existence of minimizers of the original problem, by using particular features of the minimizers of its relaxed, convex formulation. We apply the computational tools for solving the relaxed problem in four different scenarios: stable, descending, ascending and highly variable sales.

The calculations outline the existence of minimizers when the number of

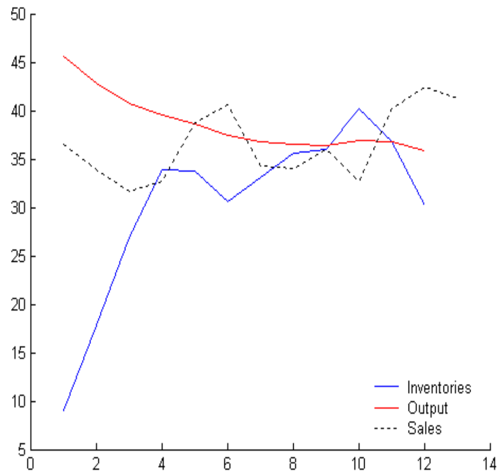

Figure 5: Optimal solution with variable sales

periods is 12 months, but not when it is 24. A possible explanation for this finding is that production and inventories decisions are only made in the short run, because firms cannot forecast sales with precision beyond 12 months. Also we found that the problem lacks of minimizers when it faces descending sales along the 12 months; a possible explanation is that it is hard for the firm to avoid the costs of decreasing sharply production or increasing rapidly its holding of inventories.

The optimal decision with stable and ascending sales consists in producing above the effective demand, allowing for accumulation of inventories; in the subsequent periods the firm diminishes its production and sales are satisfied with accumulated inventories. Thus, the firm diminishes its holding of inventories in the last period in order to avoid the higher costs this implies. We found that when sales are highly variable the production smoothing model is functional: when the demand of output is low firms produce above it, allowing the accumulation of inventories; when demand rises sharply, the firm satisfies it not by increasing production but by using its accumulated inventories, that is, firms do smooth production. This constitutes a contribution to the production smoothing model, in the sense it explains why it appear to failure when it is confronted with the data: real sales are not as variable as our theoretical sales.

For future research, this work can be extended calculating the generalized solutions when the inventory control problem lacks of minimizers. Also, price shocks and error terms can be included in the cost function, eliminated from the original objective function proposed in [29]; we suppressed those terms because its inclusion entail the analysis of stochastic optimization problems, which is beyond the scope of this paper. The reincorporation of those terms prevents us to assume any uncertainty between agents. Besides, in this work we truncated the original problem presented in [29] in order to apply the minimum principle for discrete-time problems; it would be interesting to analyze the control of inventories under non-convex polynomial cost functions as an infinite time horizon problem, using the dynamic programming algorithm and the Method of Moments.

References

[1] Akhiezer, Naum Illich (1962) *"Some questions in the theory of moments"* Translation of Mathematical Monographs, Amer. Mathematical Society.

[2] Bazaraa (1993) *"Non linear programming"* John Wiley & Sons. 2 ed.

[3] Ben-Tal, Aharon (2001) *"Lectures on Modern Convex Optimization"* TMPS, SIAM.

[4] Bertsekas (1999) *"Nonlinear Programming "* Athenea Sci.

[5] Bertsekas, Dimitri (2001) *"Dynamic Programming and Optimal Control"* Vol. I, 2nd edition. Massachusetts Institute Technology. Athena Scientific, Belmont, Massachusetts.

[6] Blinder, Alan (1986) *"Can the Production Smoothing Model of Inventory Behavior be Saved?* The Quarterly Journal of Economics, Vol. 101, No. 3, August, pp. 431-454.

[7] Bresnahan, Timothy, & Ramey, Valerie (1994) *"Output Fluctuations at the Plant Level"* The Quarterly Journal of Economics, Vol. 109, No. 3, August. pp. 593-624.

[8] Cerdá, Emilio (2001) *"Optimización Dinámica".* Universidad Complutense de Madrid. Prentice Hall, Spain.

[9] Conn, Andrew, Gould, Nicholas & Toint, Philippe (2000) *"Trust-Region Methods".* SIAM.

[10] Cooper, Russell & Haltinwager, John (1993) *"The Aggregate Implications of Machine Replacement: Theory and Evidence"* The American Economic Review, Vol. 83, No. 3, June. pp. 360-382.

[11] Curto Raul & Fialkow, Lawrence (1991) *"Recursiveness, positivity, and truncated moment problems"* Houston Journal of Mathematics Vol. 17, N.4.

[12] Escobar, Diego (1998) *"Introducción a la Economía Matemática"* Grupo Editorial Iberoamérica. Bogotá, Colombia.

[13] Fafchamps, Marcel; Gunning, Han & Oostendorp, Remco (1999) *"Inventories and Risk in African Manufacturing"* Economic Journal Vol. 110, pp. 861-893.

[14] Fletcher, Roger (1987) *"Practical methods of optimization".* John Wiley & Sons.

[15] Hall, George (2000) *"Non-Convex Costs and Capital Utilization: A Study of Production Scheduling at Automobile Asembly Plants"* Journal of Monetary Economics N. 45, pp. 681-716.

[16] Krein, M.G. & Nudel'man, A.A. (1977) *"The Markov Moment Problem and Extremal Problems"* Translations of Mathematical Monographs Vol. 50, Amer. Mathematical Society.

[17] Lasserre, Jean (2001) *"Global Optimization with polynomials and the problem of moments"* SIAM J. Optim. Vol.11, N.3, pp.796-817.

[18] Laserre, Jean (2002) *"Semidefinite programming vs lp relaxations for polynomial programming"* Annals of Operation Research Journal, Vol. 27, pp.347-360.

[19] Luenberger (1989) *"Programación lineal y no lineal"* Addison-Wesley.

[20] Mangasarian (1994) *"Nonlinear programming"* SIAM.

[21] The MathWorks Inc.(2000) *"Optimization Toolbox for Use with Matlab. User's Guide".*

[22] Meziat, René; Pedregal, Pablo & Egozcue Juan (2001) *"The Method of Moments for Non-Convex Variational Problems"* in Advances in Convex Analysis and Global Optimization, Non Convex Optimization and Its Applications Series Vol. 54, pp.371-382. Kluwer Academic Publishers.

[23] Meziat, René (2003) *"The Method of Moments in Global Optimization"* Journal of Mathematical Scienses Vol.116, pp. 3303-3324. Kluwer Academic Publishers.

[24] Meziat, René (2003) *"Analysis of Non Convex Polynomial Programs by the Method of Moments".* Nonconvex Optimization and its Application Series Vol. 74, pp. 353-372.

[25] Meziat, René (2003) *"Analysis of two dimensional non convex variational problems"* Optimization and Control with Applications. Qi, Teo and Yan eds. Publishe by Kluwer

[26] Meziat, René, Villalobos, Jorge (2005) *"Analysis of Microstructures and Phase Transition Phenomena in One-Dimensional Non-Linear Elasticity by Convex Optimization"* in press. Variational Methods and Optimization N. 3, Omeva.

[27] Meziat, René; Patiño, Diego & Pedregal, Pablo (2005) *"An Alternative Approach for Non-Linear Optimal Control Problems Based on the Method of Moments"* in press.

[28] Nesterov, Y. (2000) *"Squared functional systems and optimization problems in High Performance Optimization"* Frenk, H., K. Roos and T. Terlaky, eds., Kluwer.

[29] Ramey, Valerie (1991) *"Nonconvex Costs and the Behavior of Inventories"* The Journal of Political Economy, Vol. 99, N. 2, April, pp. 306-334.

[30] Ramey, Valerie & Vine, Daniel (2004) *"Tracking the Source of the Decline in GDP volatility: An Analysis of the Automobile Industry"* Working Paper 10384, National Bureau of Economic Research, March.

[31] Rockafellar, Tyrrell (1970) *"Convex Analysis"* Princeton University Press, Princeton, New Jersey.

[32] Shor, Naum (2000) *"Nondifferentiable Optimization and Polynomial Problems"* Kluwer.

[33] Varian, Hal R. (1996) *"Intermediate Microeconomics: A Modern Approach"* 4th edition. W.W. Norton & Company, London.

A Dirac Measure Moments Testing

Month	**Stable sales**		**Ascending sales**	
k	$(m_{1k}^*)^2 - m_{2k}^*$	$(m_{1k}^*)^3 - m_{3k}^*$	$(m_{1k}^*)^2 - m_{2k}^*$	$(m_{1k}^*)^3 - m_{3k}^*$
1	0.0000	0.0009	0.0000	0.0000
2	0.0000	0.0000	0.0000	0.0000
3	0.0000	0.0000	-0.0000	0.0000
4	0.0000	0.0000	-0.0000	0.0000
5	0.0000	0.0000	-0.0000	0.0000
6	0.0000	0.0000	0.0000	0.0000
7	0.0000	0.0000	0.0000	0.0000
8	0.0000	0.0000	-0.0000	-0.0000
9	0.0000	0.0000	-0.0000	-0.0000
10	0.0000	0.0000	0.0000	0.0000
11	0.0000	0.0000	0.0000	-0.0000
12	0.0000	0.0000	0.0000	-0.0000

Table 2: Dirac measure moments testing for Scenarios 1 and 2

Month	**Descending sales**		**Variable sales**	
k	$(m^*_{1k})^2 - m^*_{2k}$	$(m^*_{1k})^3 - m^*_{3k}$	$(m^*_{1k})^2 - m^*_{2k}$	$(m^*_{1k})^3 - m^*_{3k}$
1	0.1298	17.9	0.0000	0.0000
2	2.1283	277.1	0.0000	0.0003
3	0.8835	108.8	0.0000	0.0003
4	0.2598	30.5	0.0000	0.0000
5	0.0017	0.1	0.0000	0.0001
6	0.2569	27.1	0.0000	0.0002
7	0.4981	49.0	0.0000	0.0007
8	0.6760	62.0	0.0000	0.0000
9	0.6356	55.0	0.0000	0.0001
10	0.3694	30.9	0.0000	0.0000
11	0.0579	4.8	0.0000	0.0011
12	0.3866	33.4	0.0000	0.0001

Table 3: Dirac measure moments testing for Scenarios 3 and 4