# *Analysis of convex envelopes of polynomials and exact relaxations of non-convex variational problems*


**René Meziat - Diego Patiño**
Departamento de Matemáticas, Universidad de los Andes
Carrera 1 No 18A - 10, Bogotá, Colombia
rmeziat@uniandes.edu.co, d-patino@uniandes.edu.co


March, 2006


**Abstract**

We solve non-convex, variational problems in the form

$$\min_u I(u) = \int_0^1 f(u'(x))dx \ s.t. \ u(0) = 0, u(1) = a, \tag{1}$$

where $u \in (W^{1,\infty}(0,1))^k$ and $f : \mathbb{R}^k \to \mathbb{R}$ is a non-convex, coercive polynomial. To solve (1) we analise the convex envelope $f_c$ at the point $a$, this means to find vectors $a^1, \ldots, a^N \in \mathbb{R}^k$ and positive values $\lambda_1, \ldots, \lambda_N$ satisfying the non-linear equation

$$(1, a, f_c(a)) = \sum_{i=1}^{N} \lambda_i (1, a^i, f(a^i)). \tag{2}$$

With this information we calculate minimizers of (1) by following a proposal of B. Dacorogna in [15]. We represent the solution of (2) as the minimizer of one particular convex, optimization problem defined in probability measures which can be transformed into a semidefinite program by using necessary conditions for the solution of the multidimensional moments problem. To find explicit solutions of (1), we use a constructive, a posteriori approach which exploits the characterization of one dimensional moments on the marginal moments of a bivariate distribution. By following J.B. Lasserre's proposal on global optimization of polynomials [25], we determine conditions on $f$ and $a$ to obtain exact semidefinite relaxations of (1).



**Keywords:** calculus of variations, convex analysis, semidefinite programming, multidimensional moment problem.

# 1 Introduction

Non-convex variational problems arise in several subjects of mathematical physics like non-linear elasticity, fluid mechanics and electromagnetism. See [2, 7, 8, 10, 11, 15, 36, 41] for different applications of non-convex variational problems in solid mechanics. However, the direct methods of functional analysis do not provide any satisfactory answer for them as it is not easy to prove weak inferior semicontinuity on their functionals. Therefore, we must apply specialized techniques from non-linear analysis, usually taken from convex analysis. See [9, 15, 18, 35, 40, 44, 52]. In this paper we solve a family of non-convex variational problems by using convex relaxations in moments. This technique has been successful in global optimization of polynomials [20, 23, 24, 25, 26, 27, 29, 31, 32, 37, 38, 47], moreover it has been recently applied to the analysis of non-convex, variational problems and non-linear, optimal control problems. See [16, 30, 33, 35, 39, 42].

Let $f : \mathbb{R}^k \to \mathbb{R}$ be a coercive polynomial, this means that

$$f(x) > \alpha \, \|x\|^{\gamma} + \beta \quad \forall \, x \in \mathbb{R}^k$$

where $\gamma > 1$, $\alpha > 0$ and $\beta$ are constants. We notice that $\operatorname{co}(\operatorname{Epi}(f))$ is closed, therefore

$$\operatorname{co}(\operatorname{Epi}(f)) = \operatorname{Epi}(f_c)$$

where co stands for *convex hull* of sets in the Euclidean space $\mathbb{R}^{k+1}$, Epi stands for *epigraph* and $f_c$ represents the convex envelope of the function $f$. We notice that every point $(a, f_c(a))$ on the graph of $f_c$ can be expressed as a convex combination of points on the graph of the function $f$, that is

$$(1, a, f_c(a)) = \sum_{i=1}^{N} \lambda_i \left(1, a^i, f\left(a^i\right)\right) \tag{3}$$

where $\lambda_i > 0$ for every $i = 1, \ldots, N$ and $N \leq k+1$. See [43] for an introduction to convex envelopes of functions. In this work we solve the non-linear equation (3) called *analysis of the convex envelope of the polynomial $f$ at the point $a$*. The solution of this problem is not always unique, nevertheless we will neglect those cases with several solutions. On the other hand, B. Dacorogna has shown that the points $a^1, \ldots, a^N$ and the values $\lambda_1, \ldots, \lambda_N$ in (3) solve the following non-convex variational problem:

$$\min_u \int_0^1 f(u'(t))dt \quad s.t. \quad u(0) = 0 \quad u(1) = a \tag{4}$$

where the admissible functions $u$ belong to the Sobolev space $(W^{1,\infty}(0,1))^k$. Indeed, Formula (8) defines a minimizer for (4), see Theorem 2.6 in [[15],Chapter 5]. We represent the solution of (3) as a relaxation of (4) defined in probability measures. Probability becomes an important relaxation tool in optimization theory. It represents convex envelopes in mathematical programming, mixed strategies in game theory, generalized curves in optimal control and parametrized measures in calculus of variations. See [17, 22, 23, 24, 25, 27, 29, 32, 37, 40, 44, 50, 52] and references therein. From Jensen's inequality, we observe that the convex envelope of the polynomial $f$ at the point $a$ admits the following definition as a new optimization problem in measures:

$$f_c(a) = \min_{\mu} \int_{\mathbb{R}^k} f(s)\, d\mu(s) \tag{5}$$

where $\mu$ represents the family of all probability distributions supported in $\mathbb{R}^k$ satisfying

$$a = \int_{\mathbb{R}^k} s\, d\mu(s). \tag{6}$$

The discrete probability distribution:

$$\mu^* = \sum_{i=1}^{N} \lambda_i \delta_{a_i} \tag{7}$$

solves this optimization problem in measures. Notice that $f\left(a^i\right) = g\left(a^i\right)$ for every $i = 1, \ldots, N$, where $g(x) = f_c(a) + (x-a)\cdot y$ and $y$ is a subgradient of $f_c$ at the point $a$, i.e. $y \in \partial f_c(a)$.

**Proposition 1** *Let $f$ be a coercive, continuous function. A probability measure $\mu$ satisfying (6) is a solution of (5), if and only if it is supported in a set with the form $A_g = \left\{x \in \mathbb{R}^k : g(x) = f(x)\right\}$.*

In short, every way in which we can express the point $a$ as a convex combination of points in $A_g$ provides a different way for solving the non linear equation (3). On the other hand, every set of points $a^1, \ldots, a^N$ and values $\lambda_1, \ldots, \lambda_N$ satisfying the expression (3) determines an optimal discrete probability like (7). Since equation (3) admits only one solution, (7) is the unique solution of (5).

**Proposition 2** *The optimization problem defined in probability measures (5) is an exact relaxation of the non-convex variational problem (4) provided that $f$ be a coercive polynomial. The probability measure in (7) is its unique solution. The supporting points and probabilities of (7) solve the equation (3) and define a set of N! minimizers for (4) when they are replaced in the expression:*

$$u^*(t) = \begin{cases} a^1 t \quad \text{if} \quad 0 \le t \le \lambda_1 \\ a^1\lambda_1 + a^2\left(t-\lambda_1\right) \quad \text{if} \quad \lambda_1 \le t \le \lambda_1 + \lambda_2 \\ \vdots \\ a^1\lambda_1 + \cdots + a^{N-1}\lambda_{N-1} + a^N\left(t - \lambda_1 - \cdots - \lambda_{N-1}\right) \\ \text{if} \quad \lambda_1 + \cdots + \lambda_{N-1} \le t \le 1. \end{cases} \tag{8}$$

By using necessary conditions for the multidimensional moments problem we obtain semidefinite relaxations of (5) which can be solved by interior point algorithms. For recent results on the multidimensional moments problem see [5, 6, 13, 14, 25, 28, 45, 49]. Also see [4, 48, 51] for a good introduction to semidefinite programming and interior point methods. The approach outlined here has been also applied to obtain semidefinite relaxations of polynomial programs, see [20, 24, 25, 26, 27, 29, 31, 32, 38]. To solve the optimization problem (5), we use an a posteriori constructive approach based on the moments of marginal distributions of bivariate probability distributions. Thus, we can calculate minimizers for (4) from an exact semidefinite relaxation. By applying Lasserre's scheme of duality in semidefinite relaxations of polynomial programs in [25], we obtain a general a priori condition for obtaining exact semidefinite relaxations: the non-negative polynomial $f - g$ must be a finite sum of squares of polynomials.

The present paper is organized as follows. In Section 2 we analyze the convex envelopes of one dimensional polynomials. In Section 3, we will see the analysis of convex envelopes of two-dimensional polynomials. In Section 4 we solve particular examples of non-convex variational problems (4). Finally, in Section 5 we give some conclusions and comments about this paper and further research on non-convex, variational problems with moments and convex optimization in non linear elasticity.

## 2 Analysis of the one-dimensional situation

We deal here with the analysis of the convex envelope of one-dimensional coercive polynomials given in the general form:

$$f(t) = \sum_{i=0}^{2n} c_i t^i \qquad \text{with} \quad c_{2n} > 0. \tag{9}$$

Let us consider the convex envelope of $f$ at the point $a$, which can be expressed as:

$$f_c(a) = \min_{\mu} \int_{\mathbb{R}} f(s)\, d\mu(s) \tag{10}$$

where $\mu$ represents the family of probability measures in $\mathbb{R}$ satisfying:

$$\int_{\mathbb{R}} s\, d\mu(s) = a. \tag{11}$$

The optimal measure of (10) may have two possible forms: the first one is as a two-points supported measure

$$\text{i)} \quad \mu^* = \lambda_1 \delta_{a_1} + \lambda_2 \delta_{a_2} \tag{12}$$

therefore

$$(1, a, f_c(a)) = \lambda_1 \left(1, a_1, f\left(a_1\right)\right) + \lambda_2 \left(1, a_2, f\left(a_2\right)\right). \tag{13}$$

This situation appears where: $f_c(a) < f(a)$. The second possible form of the optimal measure $\mu^*$ is as a Dirac measure

$$\text{ii)} \quad \mu^* = \delta_a \tag{14}$$

which is found where: $f_c(a) = f(a)$. This observation comes out from a simple application of Carathedory's theorem. By using the classical solution of the truncated Hamburger moment problem [12, 21, 23], we transform the problem (10) into the semidefinite program:

$$\min_m \sum_{i=0}^{2n} c_i m_i \quad \text{s.t.} \quad H \geq 0, \quad m_0 = 1, \quad m_1 = a \tag{15}$$

where $H = (m_{i+j})_{i,j=0}^n$.

**Theorem 3** *Let $f$ be a one-dimensional, coercive polynomial with the form given in (9), then the solution $m^*$ of the corresponding semidefinite program (15) contains the algebraic moments of the optimal measure $\mu^*$ solving (10).*

**Proof.** Since $m^*$ is an optimal point of a linear objective function defined in a convex feasible set, it must be part of the boundary of such feasible set. Then, $m^*$ does not define a positive definite Hankel matrix $H^* = \left(m^*_{i+j}\right)_{i,j=0}^n$. Instead, $H^*$ is forced to be a positive semidefinite Hankel matrix. We recall from the theory of semidefinite Hankel matrices [21] that there exists an integer $k$ satisfying $0 \leq k \leq n$ such that $D_\rho > 0$ for $\rho = 0, \dots, k-1$ and $D_\rho = 0$ for every $\rho = k, \dots, n$. Here $D_\rho = \det\left[\left(m^*_{i+j}\right)_{i,j=0}^\rho\right]$ is the principal subdeterminant of $H^*$ defined by taking the first $\rho$ rows and the first $\rho$ columns of $H^*$. ($k$ is the *rank* of $H^*$ according to the definition given in [12]). By applying Fisher's theorem [46], we can see that there exists a unique discrete probability measure supported in $k$ points:

$$\mu^* = \sum_{i=1}^k \lambda_i \delta_{a_i}$$

whose first $2n$ moments are the values $m^*_0, \dots, m^*_{2n-1}$ respectively, and its $(2n+1)$-th moment does not exceed the value $m^*_{2n}$. Therefore

$$\int_{\mathbb{R}} f(s)\, d\mu^*(s) \leq \sum_{i=0}^{2n} c_i m^*_i \leq \int_{\mathbb{R}} f(s)\, d\mu(s) \tag{16}$$

for every probability measure $\mu$ in $\mathbb{R}$ satisfying (11). The left side inequality in (16) comes out from the positivity of the leader coefficient $c_{2n}$ of the polynomial $f$. The right side inequality in (16) comes out from the fact that the moments of every positive measure form a positive semidefinite Hankel matrix.

■

Now, we apply this result to the analysis of one-dimensional, non-convex, variational problems (4) and we will see that we can obtain minimizers for (4) from their convex relaxations (15).

**Corollary 4** *The Hankel matrix $H^* = \left(m^*_{i+j}\right)_{i,j=0}^n$ constructed by using the optimal values of the program (15) only has rank one or two. Moreover, its rank is determined by the convexity of $f$ at the point $a$.*

If $H^* = (m^*_{i+j})^n_{i,j=0}$ has rank one, its entries are the algebraic moments of the Dirac measure $\mu^* = \delta_a$. We can determine $\mu^* = \delta_a$ by taking $m^*_1 = a$ as its supporting point. Otherwise $H^*$ has rank two. In this case we can obtain the optimal measure $\mu$ by solving the second degree algebraic equation:

$$P(t) = \begin{vmatrix} m^*_0 & m^*_1 & m^*_2 \\ m^*_1 & m^*_2 & m^*_3 \\ 1 & t & t^2 \end{vmatrix} = 0 \tag{17}$$

which has two real roots $a_1$, $a_2$, where $a_1 < a < a_2$. See [1]. By taking:

$$\lambda_1 = \frac{a_2 - a}{a_2 - a_1} \quad and \quad \lambda_2 = \frac{a - a_1}{a_2 - a_1} \tag{18}$$

we obtain:

$$\mu^* = \lambda_1 \delta_{a_1} + \lambda_2 \delta_{a_2}.$$

Summarizing, by solving the convex program (15) we obtain the optimal measure $\mu^*$ of (10). In addition, the support and the probabilities of $\mu^*$ determine the analysis of the convex envelope of the polynomial (10) at the point $a$. Thus, in one dimensional cases, the semidefinite program (15) is always an exact relaxation of the corresponding variational problem (4).

**Remark 5** *In order to calculate a three-points supported measure on the real line from its moments, we must increase the determinant in (17), so we obtain a third degree algebraic equation with three real roots. This procedure was proposed in [1] and it will be useful later for the two-dimensional case.*

# 3 Analysis of two-dimensional polynomials

To analyse convex envelopes of two-dimensional, coercive polynomials

$$f(x, y) = \sum_{0 \le i+j \le 2n} c_{i,j} x^i y^j \tag{19}$$

in one particular point $a = (a_1, a_2)$ on the plane, we must use a different approach. The reason relies on the fact that we can not properly characterize the cone of two indexed vectors $(m_{i,j} : 0 \le i + j \le 2n)$ of moments of bivariate positive measures by using a single linear matrix inequality. See [5, 6, 13, 14, 28, 45, 49] for recent accounts on the characterization of multidimensional moments and [24, 25, 26, 27, 29] for their implications in global optimization of polynomials. A necessary condition for $(m_{i,j} : 0 \le i + j \le 2n)$ to be a vector of moments, is that its entries form a semidefinite positive quadratic form like $(m_{i+i',j+j'})_{0 \le i+j \le n,\, 0 \le i'+j' \le n}$. Thus, we have the following result.

**Proposition 6** *Every semidefinite program:*

$$\begin{aligned} \min_m \sum_{0 \le i+j \le 2n} & c_{i,j} m_{i,j} \\ \text{s.t. } & (m_{i+i',j+j'})_{0 \le i+j \le n',\, 0 \le i'+j' \le n'} \ge 0 \\ \text{with } & \quad m_{0,0} = 1;\ m_{1,0} = a_1;\ m_{0,1} = a_2; \quad n' \ge n \end{aligned} \tag{20}$$

*is a lower bound for the problem (5) with $k = 2$. Hence, (20) is a lower bound of the non-convex variational problem (4).*

Now we establish sufficient, a posteriori conditions to make (20) an exact relaxation of (4) in the two dimensional case, i.e. $k = 2$.

**Proposition 7** *When the values $\{m^*_{i,j} : 0 \leq i + j \leq 2n'\}$ solve (20) and they are the moments of a bivariate probability distribution, then the semidefinite program (20) is an exact relaxation of the corresponding variational problem (4) and the values $m^*_{i,j}$ must be the moments of the probability measure $\mu^*$ that solves (5).*

Thus, to obtain exact relaxations of (4) we must solve the semidefinite program (20) and verify if we obtain a valid set of moments $m^*_{i,j}$. If we can achieve this aim, we obtain the moments of the probability measure $\mu^*$ that solves the non-linear equation (3). We propose a particular method to do this. It consists in trying the construction of a measure $\mu^*$ from the values $m^*_{i,j}$. Since the optimal measure (7) that we search for is supported in three points at most, we calculate first the marginal distributions $\mu^*_X$ and $\mu^*_Y$ by applying the procedure for one dimensional problems on the *marginal* values $m^*_{i,0}$ and $m^*_{0,j}$. In general, it is not possible to recover a bivariate probability distribution from its marginal distributions. Nevertheless, we can obtain $\mu^*$ from its marginal distributions owed to its extremely simple form. Next, as $\mu^*$ was obtained by using only the marginal values $m^*_{i,0}, m^*_{0,j}$, we must check that *all* the bivariate moments of the constructed distribution $\mu^*$ coincide with their corresponding values $m^*_{i,j}$ obtained in (20). When this verification procedure is right, the constructed measure $\mu^*$ must have the form (7). Hereafter, we can use its support and its probabilities to determine a set of minimizers of the variational problem (4). If this procedure works well, we can conclude that (20) is an exact relaxation of (4). Now we establish a priori conditions to obtain exact semidefinite relaxations of (4). We present an analysis based on the dual program of the semidefinite relaxation (20) following the same approach proposed by J.B. Lasserre in [25] for global optimization of polynomials. We also need a *supporting,* linear function $g(x) = f_c(a) + (x - a) \cdot y$ with $y \in \partial f_c(a)$. Notice that $g(a) = f_c(a)$ and $g \leq f$.

**Theorem 8** *If there exists a $y \in \partial f_c(a)$ such that the positive polynomial $f - g$ can be expressed as a sum of squares of polynomials whose degrees do not exceed $n'$, then we can guarantee that (20) is an exact relaxation of (4).*

**Proof.** To prove the strict feasibility of the semidefinite program (20), we take the moments of any probability distribution with continuous density in $\mathbb{R}^2$ whose marginal first order moments are the values $a_1$ and $a_2$. See [4, 14, 25, 29, 48]. The dual form of the semidefinite program (20) is

$$\begin{gathered} \max -\gamma_{0,0} - 2a_1\gamma_{1,0} - 2a_2\gamma_{0,1} \quad s.t. \\ \langle A_{i,j}, \Gamma \rangle = c_{i,j} \quad for \quad 0 \leq i + j \leq 2n' \quad and \quad \Gamma \geq 0 \end{gathered} \tag{21}$$

where $\Gamma = (\gamma_{i+i',j+j'})_{0\leq i+j\leq n', 0\leq i'+j'\leq n'}$, $\langle\cdot\rangle$ stands for the Frobenius product and every matrix $A_{i,j}$ is full of zeros, but its $i$-$j$ entries where there are ones. The $i$-$j$ entries are the positions of the function $x^i y^j$ in the quadratic form $ZZ^t$ with $Z = (x^i y^j)_{0\leq i+j\leq n'}$. See [37, 25]. Without lost of generality we can assume that $f$ does not have linear nor constant terms, so if $f - g = \sum_{j=0}^{l} q_j^2$ we can take $\Gamma = \sum_{j=0}^{l} q_j q_j^t$ where $q_j(x,y) = q_j \cdot Z$. Finally, notice that $\Gamma$ is a factible point for the dual program (21) whose value in the objective function is $f_c(a)$. ■

If the problem does not fulfill the condition of this theorem, we still can fit the problem into a compact set and use recent proposals on the characterization of multidimensional moments of measures on compact semi-algebraic sets. This would entail one additional linear matrix inequality constraint in the semidefinite program (20). In this case, we can use the fact that positive polynomials like $f - g$ can be approximated by a sequence of sums of squares of polynomials on particular compact sets. This fact guarantees that we can obtain an increasing sequence of lower bounds of (4) which should converge to the optimal value of (4). See [25].

# 4 Examples

In this section we calculate minimizers of non-convex, variational problems like (4). We used the routines described in [19] to solve every semidefinite program.

## 4.1 One-dimensional, non-convex variational problems

We solve (4) with $k = 1$, the non-convex, eight degree polynomial integrand:

$$f(t) = t^8 - t^7 - 3t^6 + 2.2t^5 + 3t^4 - t^3 - t^2 - 0.1t + 1.4$$

and the boundary condition $a = -0.5$. The optimum measure is

$$\mu^* = 0.6445\delta_{-1.0840} + 0.3555\delta_{0.5589}. \tag{22}$$

Thus, there are two minimizers in $W^{1,\infty}(0,1)$, one of them is:

$$u^*(x) = \begin{cases} -1.084x & 0 \leq x \leq 0.6445 \\ 0.5589(x - 0.6445) - 0.6986 & 0.6445 < x \leq 1. \end{cases}$$

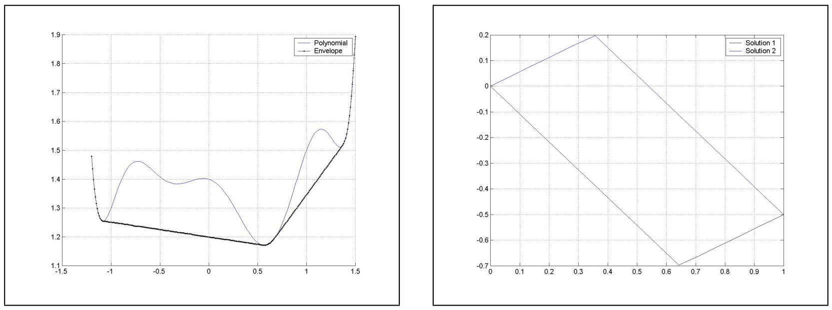
Figure 1(a) shows the convex envelope of $f$ and Figure 1(b) shows the minimizers of (4). By taking the boundary condition $a = 1$, we obtain the optimal measure:

$$\mu^* = 0.5209\delta_{0.6638} + 0.4791\delta_{1.3655} \tag{23}$$

and the couple of minimizers shown in Figure 2. The explicit expression for one of them is:

$$u^*(x) = \begin{cases} 0.6638x & 0 \leq x \leq 0.5209 \\ 1.3655(x - 0.5209) + 0.3458 & 0.5209 < x \leq 1. \end{cases}$$

For $a = 1.5$ there is only one minimizer: $u^*(x) = 1.5x$ because the optimal measure of (5) is the Dirac measure $\delta_{1.5}$.




(a) One dimensional non convex polynomial and its convex envelope

(b) Minimizers $u^*$ when $a = -0.5$

Figure 1: Solution of a one dimensional non convex variational problem

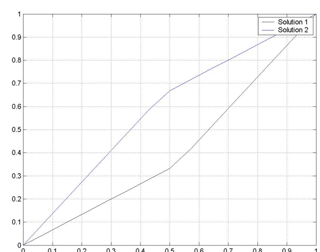



Figure 2: Minimizers $u^*$ when $a = 1$

## 4.2 Two-dimensional, non-convex variational problems

Now we will see the particular features of the method when it is applied on two-dimensional cases.

### 4.2.1 2D Example

To analyse the convex envelope of the eight degree polynomial

$$f(x,y) = (x^2+y^2+1)((x-1)^2+(y-1)^2)((x-2)^2+(y+1)^2)((x-1)^2+(y+1)^2+1) \tag{24}$$

at the point $a = (1.9, -0.8)$, we solve the corresponding semidefinite program (20) with $n' = 4$. We use the values $m^*_{i,0}$, $m^*_{0,j}$ to construct the marginal measures

$$\begin{aligned}\mu_X^* &= 0.1\delta_1 + 0.9\delta_2 \\ \mu_Y^* &= 0.1\delta_1 + 0.9\delta_{-1}\end{aligned}$$

which provide the optimal bivariate probability measure

$$\mu^* = 0.1\delta_{(1,1)} + 0.9\delta_{(2,-1)}. \tag{25}$$

Figure 3(a) shows the non-convex surface of the polynomial (24), Figure 3(b) shows the support of the optimal measure $\mu^*$ given in (25). We must check that all moments of the measure $\mu^*$ given in (25) coincide with the optimal values obtained in the program (20). The results of this verification procedure are shown in Table 1 where $\widetilde{m_{i,j}} = \iint x^i y^j d\mu^*$. In this case, we obtain an exact semidefinite relaxation of (4) with the polynomial (24) and the boundary condition $a = (1.9, -0.8)$. The minimizers of this problem in $(W^{1,\infty}(0,1))^2$ are shown in Figure 4. One of them is:

$$u^*(x) = \begin{cases} (1,1)x & 0 \le x \le 0.1 \\ (x-0.1)(2,-1) + (0.1, 0.1) & 0.1 < x \le 1. \end{cases} \tag{26}$$

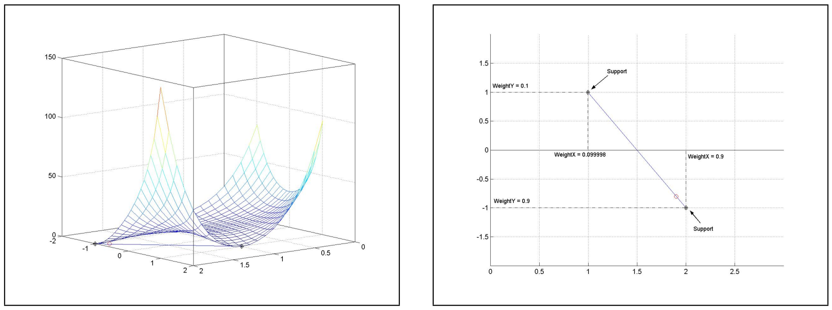
| Moment | $\widetilde{m_{i,j}}$ | $m_{i,j}^*$ | $\lvert\widetilde{m_{i,j}} - m_{i,j}^*\rvert$ | Moment | $\widetilde{m_{i,j}}$ | $m_{i,j}^*$ | $\lvert\widetilde{m_{i,j}} - m_{i,j}^*\rvert$ |
|---|---|---|---|---|---|---|---|
| $m_{1,1}$ | -0.17 | -0.17 | 1.6e-009 | $m_{3,3}$ | -1.8 | -1.8 | 2.1e-008 |
| $m_{2,1}$ | -0.7 | -0.7 | 1.9e-009 | $m_{4,3}$ | -3.9 | -3.9 | 4.7e-008 |
| $m_{3,1}$ | -1.8 | -1.8 | 8.6e-009 | $m_{3,4}$ | 2.5 | 2.5 | 2.7e-008 |
| $m_{1,2}$ | 0.9 | 0.9 | 5.9e-009 | $m_{7,1}$ | -34 | -34 | 1.8e-007 |
| $m_{2,2}$ | 1.4 | 1.4 | 1e-008 | $m_{6,2}$ | 17 | 17 | 1.4e-007 |
| $m_{1,3}$ | -0.17 | -0.17 | 1.5e-009 | $m_{5,3}$ | -8.2 | -8.2 | 9.9e-008 |
| $m_{4,1}$ | -3.9 | -3.9 | 2.2e-008 | $m_{4,4}$ | 4.6 | 4.6 | 5.4e-008 |
| $m_{3,2}$ | 2.5 | 2.5 | 1.9e-008 | $m_{2,4}$ | 1.4 | 1.4 | 1.4e-008 |
| $m_{2,3}$ | -0.7 | -0.7 | 8.2e-009 | $m_{1,5}$ | -0.17 | -0.17 | 2.3e-009 |
| $m_{1,4}$ | 0.9 | 0.9 | 7.1e-009 | $m_{2,5}$ | -0.7 | -0.7 | 1e-008 |
| $m_{5,1}$ | -8.2 | -8.2 | 4.7e-008 | $m_{3,5}$ | -1.8 | -1.8 | 2.7e-008 |
| $m_{6,1}$ | -17 | -17 | 9.5e-008 | $m_{1,6}$ | 0.9 | 0.9 | 2.8e-009 |
| $m_{4,2}$ | 4.6 | 4.6 | 3.7e-008 | $m_{2,6}$ | 1.4 | 1.4 | 7.7e-009 |
| $m_{5,2}$ | 8.9 | 8.9 | 7.2e-008 | $m_{1,7}$ | -0.17 | -0.17 | 4e-009 |

Table 1: Verification procedure

### 4.2.2 2D Example

We analyse the polynomial:

$$f(x,y) = (x^2+y^2)((x-1)^2+(y-1)^2)((x-2)^2+(y+1)^2)((x-1)^2+(y+1)^2+1) \tag{27}$$




(a) Non convex surface (b) Support of the optimal measure

Figure 3: Analysis of a convex envelope

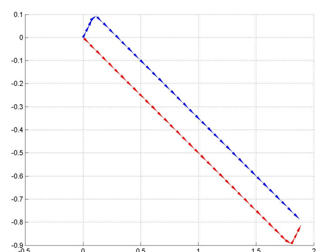


Figure 4: Minimizers $u^*$ for $(1.9, -0.8)$

in the point $a = (0.9, 0.1)$. After solving the program (20) with $n' = 4$ and the coefficients of (27), we obtain the following marginal measures

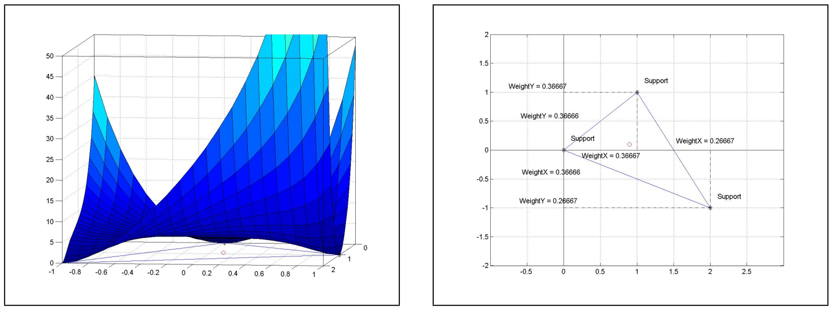
$$\begin{aligned} \mu_X^* &= 0.2667\delta_2 + 0.3667\delta_1 + 0.3666\delta_0 \\ \mu_Y^* &= 0.2667\delta_{-1} + 0.3667\delta_1 + 0.3666\delta_0 \end{aligned}$$

and the optimal bivariate measure:

$$\mu^* = 0.2667\delta_{(2,-1)} + 0.3667\delta_{(1,1)} + 0.3666\delta_{(0,0)}$$

shown in Figure 5(b). In this example, the verification procedure of comparing the moments of $\mu^*$ against the optimal values of the program (20) has been satisfactory, but it is not shown here. In this case we obtain an exact relaxation of (4). Figure 6 shows the minimizers in $(W^{1,\infty}(1,0))^2$. We show the explicit expression obtained for one of them:

$$u^*(x) = \begin{cases} (2,-1)x & 0 \leq x \leq 0.2667 \\ (1,1)(x - 0.2667) + (0.5334, -0.2667) & 0.2667 \leq x \leq 0.6334 \\ (0.9001, 0.1) + (0,0)(x - 0.6334) & 0.6334 \leq x \leq 1. \end{cases} \quad (28)$$




(a) Non convex surface

(b) Support of the optimal measure

Figure 5: Analysis of a convex envelope

#### 4.2.3 2D Example

When we analyse the convex envelope of the eight degree polynomial

$$f(x,y) = ((x+2)^2+y^2)((x-1)^2+(y-1)^2)(x^2+(y+1)^2)((x-0.1)^2+(y+1)^2+1) \tag{29}$$

at the point $a = (-0.2, 0)$ we obtain the optimal measure

$$\mu^* = 0.28\delta_{(-2,0)} + 0.36\delta_{(0,-1)} + 0.36\delta_{(1,1)}$$

shown in Figure7(b). The $(W^{1,\infty}(0,1))^2$ minimizers are shown in Figure 8. Here we describe one of them:

$$u^*(x) = \begin{cases} (-2,0)x & 0 \le x \le 0.28 \\ (0,-1)(x-0.28) + (-0.56,0) & 0.28 \le x \le 0.64 \\ (-0.56,-0.36) + (1,1)(x-0.64) & 0.64 \le x \le 1. \end{cases} \tag{30}$$

#### 4.2.4 2D Example

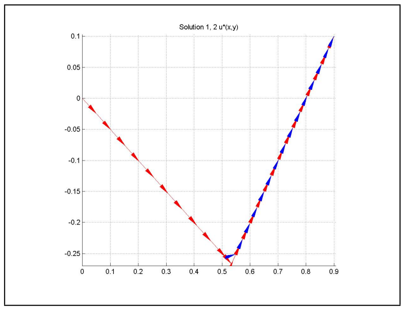
Here we solve the variational problem (4) where $f$ is the sixth degree, two variables polynomial:

$$f(x,y) = ((x-0.1)^2+(y+0.1)^2)((x-1)^2+(y-1)^2)((x-2)^2+(y+1)^2) \tag{31}$$

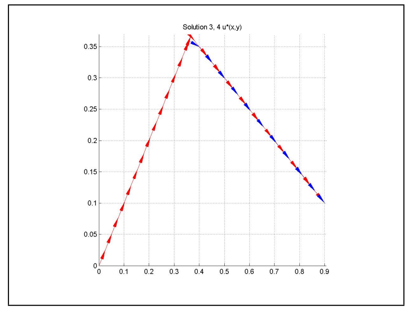
and the boundary condition is $a = \left(\frac{1}{2}, 0\right)$. In this case, we obtain the optimal measure:

$$\mu^* = 0.1205\delta_{(2,-1)} + 0.19\delta_{(1,1)} + 0.6895\delta_{(0.1,-0.1)}$$

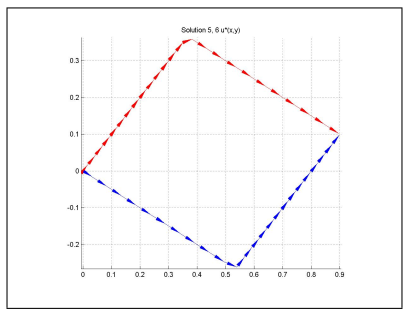
whose support is illustrated in Figure 9 beside six different minimizers in $(W^{1,\infty}(0,1))^2$.




(a) Minimizers when $a = (0.9, 0.1)$




(b) Minimizers when $a = (0.9, 0.1)$




(c) Minimizers when $a = (0.9, 0.1)$

Figure 6: Minimizers of a $2D$ non convex variational problem.

### 4.2.5 2D Example

For the sixth degree polynomial:

$$f(x, y) = ((x-1)^2 + (y+1)^2)((x-2)^2 + (y-2)^2)((x-0.3)^2 + (y+0.5)^2)$$

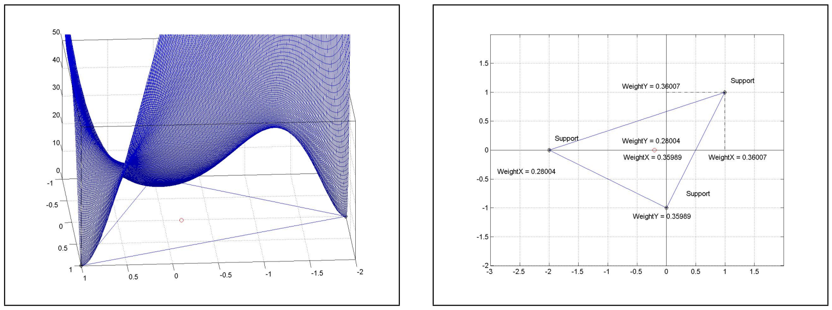
and the boundary condition $a = (1.2, 0)$ we obtain the optimal measure

$$\mu^* = 0.3074\delta_{(2,2)} + 0.5439\delta_{(1,-1)} + 0.1487\delta_{(0.3,-0.5)}$$

and the $(W^{1,\infty}(0, 1))^2$ minimizers shown in Figure 10.

# 5 Concluding Remarks

In this paper we have presented a practical way for solving non-convex, variational problems in the form (4) by using semidefinite relaxations. We propose a constructive approach which allows us to calculate minimizers of (4). To overcome the difficulties of the characterization of multidimensional algebraic




(a) Non convex surface (b) Support of the optimal measure

Figure 7: Analysis of a convex envelope

moments, we use well known procedures to determine one-dimensional measures from its moments on the marginal distributions of bivariate probability distributions. In addition, we determine general conditions on $f$ and $a$ which guarantee the applicability of our proposal. Although there are qualitative, important differences between the one-dimensional case and the multi-dimensional cases, in practice we can solve many non-convex variational problems in the form (4) by reducing them to exact semidefinite relaxations which provide sets of minimizers. On the other hand, the analysis of convex envelopes of functions with respect to other kinds of convexities like polyconvexity, rank one convexity and quasiconvexity is crucial for a better understanding of oscillatory phenomena in non-linear elasticity [7, 9, 41]. Recent advances on this matter with moments and global optimization techniques can be found in [3, 34].

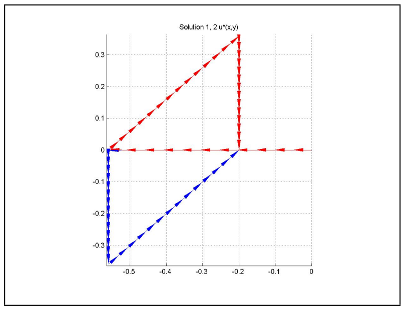
**Acknowledgements** CNRS France, Universidad de Castilla la Mancha, Centro Internacional de Métodos Numéricos de Barcelona, Agencia Española de Cooperación Internacional, Colciencias 1204-05-13627, Facultad de Ciencias de la Universidad de los Andes.

# References

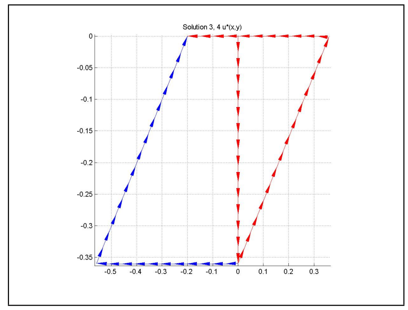
[1] N.I. Akhiezer. *Some questions in the theory of moments.* Translations of mathematical monographs. Amer. Mathematical Society, 1962.

[2] G. Allaire. *Shape Optimization by the Homogenization Method.* Springer Verlag, 2002.

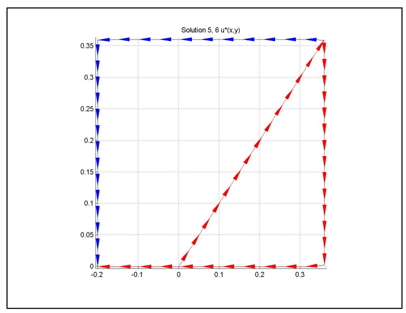
[3] G. Beliakov and R. Meziat. Estimation of minimizers of non convex variational problems by global optimization techniques. 2005.




(a) Minimizers when $a = (-0.2, 0)$




(b) Minimizers when $a = (-0.2, 0)$




(c) Minimizers when $a = (-0.2, 0)$

Figure 8: Minimizers of a $2D$ non convex variational problem.


[4] A. Ben-Tal and A. Nemirovski. *Lectures on Modern Convex Optimization: Analysis, Algorithms, and Engineering Applications*. Series on Optimization 2. Soc for Industrial & Applied Math, Aug 2001.

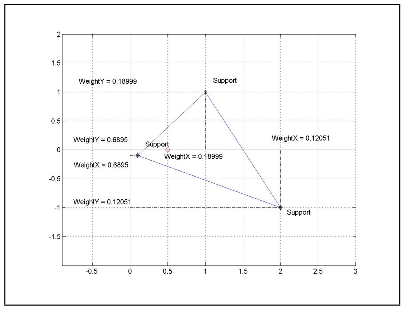
[5] C. Berg. The multidimensional moment problem and semigroups. volume 37 of *Moments in mathematics . (San Antonio, Tex., 1987)*, pages 110–124, Providence, RI, 1987. Proc. Sympos. Appl. Math., Amer. Math. Soc.

[6] G. Cassier. Problème des moments sur un compact de $\mathbb{R}$. et décomposition de polynômes à plusieurs varaibles. *J. Funct. Anal.*, 58:254–266, 1984.

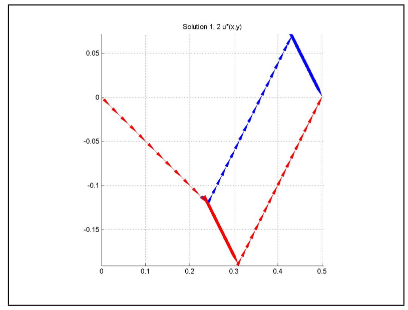
[7] A. Cherkaev. *Variational Methods for Structural Optimization*. Springer Verlag, 2000.

[8] M. Chipot. Numerical analysis of oscillations in non convex problems. *Numerische Mathematik*, 59:747–767, 1991.

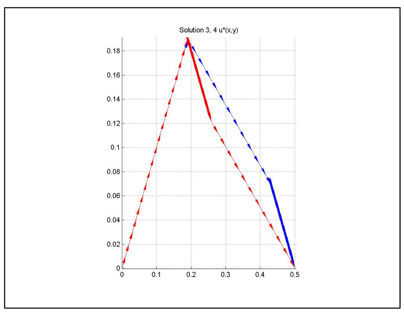
[9] M. Chipot. *Elements of nonlinear analysis*. Birkhäuser Verlag, 2000.



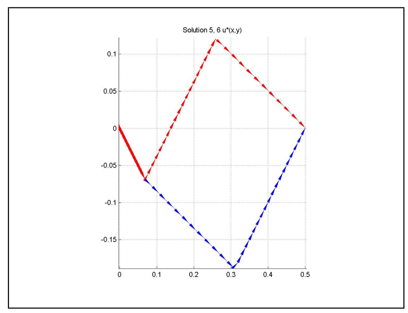

(a) Support of the optimal measure




(b) Minimizers when $a = (0.5, 0)$




(c) Minimizers when $a = (0.5, 0)$




(d) Minimizers when $a = (0.5, 0)$

Figure 9: Minimizers of a $2D$ non convex variational problem.


[10] M. Chipot, C. Collins, and D. Kinderlehrer. Numerical analysis of oscillations in multiple well problems. *Numerische Mathematik*, 70(3):259 – 282, Apr 1995.

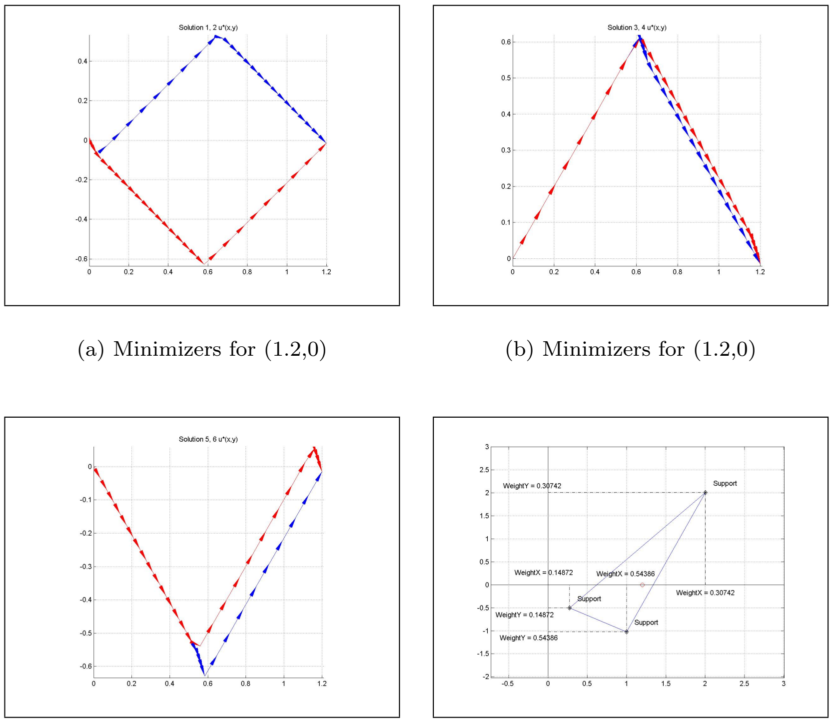
[11] M. Chipot and D. Kinderlehrer. Equilibrium configurations of crystals. *Archive for Rational Mechanics and Analysis*, 103:237–277, 1988.

[12] R. Curto and L.A. Fialkow. Recursiveness, positivity, and truncated moment problems. *Houston Journal of Mathematics*, 17(4):603, 1991.

[13] R. Curto and L.A. Fialkow. *Solution of the Truncated Complex Moment Problem for Flat Data*, volume 119 of *Memoirs of the American Mathematical Society*. American Mathematical Society, 1996.

[14] R. Curto and L.A. Fialkow. The truncated complex k-moment problem. *Trans. Amer. Math. Soc.*, 352(6):2825–2855, 2000.

[15] B. Dacorogna. *Direct Methods in the Calculus of Variations (Applied Mathematical Sciences)*. Springer-Verlag, May 1989.




(a) Minimizers for (1.2,0)

(b) Minimizers for (1.2,0)

(c) Minimizers for (1.2,0)

(d) Support of the optimal measure

Figure 10: Non-convex variational problem

[16] R. Meziat J. Egozcue and P. Pedregal. The method of moments for non-convex variational problems. In P. Pardalos and N. Hadjisavvas, editors, *Advances in Convex Analysis and Global Optimization*, volume 54 of *Nonconvex Optimization and its Applications Series*, pages 371–382. Kluwer, 2001.

[17] J. Frenk et al. On borel probability measures and noncooperative game theory. *Optimization*, 54, No 1:81–101, Taylor and Francis, 2004.

[18] J.Jost et al. *Calculus of Variations (Cambridge Studies in Advanced Mathematics)*. Cambridge University Press, January 1999.

[19] P. Gahinet et al. Lmi control toolbox for use with matlab. The MathWorks Inc, 1995.

[20] D. Henrion and J.B. Lasserre. GloptiPoly: Global optimization over polynomials with Matlab and SeDuMi. *ACM Transactions on Mathematical Software*, 29(2):165–194, jun 2003.

[21] I. Iohvidov. *Hankel and Toeplitz Matrices and Forms*. Springer Verlag, Sep 1982.

[22] S. Karlin. *Tchebycheff systems: With applications in analysis and statistics*. Pure and applied mathematics. Interscience Publishers, 1966.

[23] M.G. Krein. *The Markov Moment Problem and Extremal Problems: Ideas and Problems of P.L. Chebyshev and A.A. Markov and Their Further Development*. Translations of mathematical monographs. American Mathematical Society, Jun 1977.

[24] J. B. Lasserre. Polynomial programming: Lp-relaxations also converge.

[25] J. B. Lasserre. Global optimization with polynomials and the problem of moments. *SIAM Journal on Optimization*, 11(3):796 – 817, 2001.

[26] J. B. Lasserre. Polinomials nonnegative on a grid and discrete optimization. *Trans. Amer. Math. Soc.*, 354:631–649, 2002.

[27] J. B. Lasserre. Semidefinite programming vs. lp relaxations for polynomial programming. *Mathematics of Operations Research*, 27(2):347–360, 2002.

[28] M. Laurent. Revisiting two theorems of curto and fialkow on moment matrices. 2004.

[29] M. Laurent and D. Jibetean. Semidefinite approximations for global unconstrained polynomial optimization. 2005.

[30] J.Egozcue R. Meziat and P. Pedregal. From a non-linear non-convex formulation to a linear convex formulation of non-convex variational problems. *Journal of Applied Mathematics and Optimization*, 47:27–44, 2003.

[31] R. Meziat. Analysis of non convex polynomial programs by the method of moments. In Christodoulos A. Floudas and P. M. Pardalos, editors, *Frontiers in Global Optimization*, volume 74 of *Nonconvex Optimization and its Applications*, pages 353–372. Springer-Verlag, 2003.

[32] R. Meziat. The method of moments in global optimization. *Journal of Mathematical Sciences*, 116(3):3303–3324, Jul 2003.

[33] R. Meziat. Analysis of two dimensional nonconvex variational problems. In Qi. Liqun T. Koklay and Y. Xiaoqi, editors, *Optimization and Control with Applications*, volume 96 of *Applied Optimization*. Springer-Verlag, 2005.

[34] R Meziat and D. Patiño. Poly-convexification of matricial functions with moments. 2005.

[35] R. Meziat and J. Villalobos. A new perspective in microstructures and phase transition phenomenae in one-dimensional non-linear elasticity. 2004.

[36] S. Müller. Variational models for microstructures and phase transitions. Lecture notes #2, Max-Planck-Institut für Mathematik in den Naturwissenschaften, 1998.

[37] Y. Nesterov. Squared functional systems and optimization problems. High Performance Optimization. Kluwer, 2000.

[38] P.A. Parrilo. Semidefinite programming relaxations for semialgebraic problems. *Mathematical Programming*, 96(2):293 – 320, may 2003.

[39] R Meziat D. Patiño and P. Pedregal. An alternative approach for non linear optimal control problems based on the method of moments. 2004.

[40] P. Pedregal. *Parametrized Measures and Variational Principles*, volume 30 of *Progress in Nonlinear Differential Equations and Their Applications*. Birkhauser, 1997, May 1997.

[41] P. Pedregal. *Variational Methods in Nonlinear Elasticity*. Soc for Industrial & Applied Math, Mar 2000.

[42] C. Rivera. *Problemas variacionales no convexos con restricciones en la derivada*. MSc. Thesis Universidad de los Andes Bogota, 2004.

[43] T.R. Rockafellar. *Convex analysis*. Princeton University Press, Princeton, New Jersey, 1970.

[44] T. Roubicek. *Relaxation in Optimization Theory and Variational Calculus*, volume 4 of *De Gruyter Series in Nonlinear Analysis and Applications*. Walter de Gruyter Inc, Jun. 1997.

[45] K. Schmudgen. The k-moment problem for compact semi-algebraic sets. *Math. Ann.*, 289:203–206, 1991.

[46] J. A. Shohat and J. D. Tamarkin. The problem of moments. *Mathematical Surveys*, (1), 1943.

[47] N. Z. Shor. Quadratic optimization problems. *Tekhnicheskaya Kibernetika*, 1:128–139, 1987.

[48] L. Vandenberghe and S. Boyd. Semidefinite programming. *SIAM review*, 38, 1996.

[49] F.H. Vasilescu. *Spectral measures and moments problems*. Spectral theory and its applications. Theta, 2003.

[50] C.F. Wen and S.Y. Wu. Duality theorems and algorithms for linear programming in measure spaces. *Journal of Global Optimization*, 30, No 2-3:207 – 233, 2004.

[51] M. W. Wright. The interior points revolution. Technical report, Annual meeting of the AMS, Phoenix, Jan 2004.

[52] L.C. Young. *Lectures on the Calculus of Variations and Optimal Control Theory*. Chelsea Pub Co, 2nd edition, 1980.