THE METHODS OF MOMENT FOR SOME NONLOCAL VARIATIONAL PRINCIPLES

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1. Introduction. In this paper we tackle with a nonlocal variational problem of the form

$$m = \inf_{u\in\mathcal{A}} I(u), \quad \text{with } I(u) = \iint_{J\times J} W(x, y, u(x), u(y), u'(x), u'(y))\, dx\, dy, \tag{P}$$

where $W : J \times J \times \mathbb{R}^4 \to \mathbb{R}$ is continuous in the variables u and u', and measurable on x and y, J is an open interval in $\mathbb{R}$, $\mathcal{A} = \left\{u \in W^{1,p}(J) : u - u_0 \in W_0^{1,p}(J)\right\}$ for some $u_0 \in W^{1,p}(J)$ such that $I(u_0) < +\infty$ and $p > 1$.

Problems of this type appears in some nonlocal models with long range interaction energies, such that phase transitions problems, ferromagnetism or fracture mechanics (cf. [**?**, **?**, **?**, **?**, **?**]). Tipically, as usual in the context of calculus of variations, in absence of certain properties for W which guarantees the weak lower semicontinuity of these functions, the minimum of (P) may not exists and minimizing sequences develop oscillations in such a way that their weak limits are not minimizers.

One of the main difficulties of this problem is that the questions about weak lower semicontinuity and relaxation are not completely understood. In [**?**], Bevan and Pedregal have found a necessary and sufficient condition for the weak lower semicontinuity of I in the homogeneous case, that is, when W only dependes on the variables u', being those techniques useless in the general case. In that work, they proved that

$$I(u) = \iint_{J\times J} W(u'(x), u'(y))\, dx\, dy$$

is weak lower semicontinuous if and only if the symmetric part of W is separately convex.[1] Moreover, they show that in general, the separetely convex hull of W is not the corresponding relaxed integrand which provides the weak lower semicontinuity of I. Indeed, as it will be shown afterwards, it is not clear whether it can be defined.

The only possible way in which relaxation can be performed is in terms of Young measures, which provides a framework where existence, under coerciveness hypothesis, is ensured and

[1] The symmetric and anti-symmetric part of W, W^+ and W^-, are defined by

$$W^{\pm}(r, s) = \frac{W(r, s) \pm W(s, r)}{2}.$$

Note that $W = W^+ + W^-$ and Fubini's theorem implies

$$\iint_{J\times J} W^-(u'(x), u'(y))\, dx\, dy = 0.$$

the behaviour of minimizing sequences can be codified through them. This relaxation was carried on in [?], where it was proven that if W satisfies

$$\begin{aligned} c\left(|u|^p+|v|^p+|r|^p+|s|^p\right)+m(x,y) &\leq W(x,y,u,v,r,s), \\ W(x,y,u,v,r,s) &\leq Cc\left(|u|^p+|v|^p+|r|^p+|s|^p\right)+M(x,y), \end{aligned} \tag{1}$$

for $0<c<C$, and $m, M \in L^1(J\times J)$, then the problem

$$\tilde{m}=\min_{\nu\in\tilde{\mathcal{A}}}\tilde{I}(\nu), \quad \text{with } \tilde{I}(\nu)=\iint_{J\times J}\iint_{\mathbb{R}^2} W(x,y,u(x),u(y),r,s)\,d\nu(r)\,d\nu(s)\,dx\,dy, \tag{$\tilde{P}$}$$

where $\tilde{\mathcal{A}}$ is the set of probability measures $\{\nu_x\}_{x\in J}$ such that

$$\int_J\int_{\mathbb{R}}|r|^p\,d\nu_x(r)<\infty, \quad u'(x)=\int_{\mathbb{R}} r\,d\nu_x(r), \quad u\in\mathcal{A},$$

admits a solution ν_0, $\tilde{m}=m$ and

$$\lim_{j\to\infty} I(u_j)=\tilde{I}(\nu_0),$$

where $\{u_j\}$ is a minimizing sequence for (P).

In the common problems where only one measure takes part, the interaction between the convex hull of the integrand and the measure solution allow to get this one, and therefore the behaviour of minimizing sequences. However, since the special circunstancies commented above, the information coming from the different convex hulls do not give us anything about the optimal measure. In [?] quite complicated necessary optimality conditions are given in order to obtain the solution for $(\tilde{P})$. As far as we know, this is the only work in which a method for obtaining solution of this type of problems has perfomed.

Let us assume that W is a coercive $2n$ degree-bidimensional polynomial, which implies that $(\tilde{P})$ has a solution, that is

$$W(r,s)=\sum_{0\leq i+j=2n} c_{i,j}r^is^j$$

and let us write in the form

$$W(r,s)=c_{0,0}+\sum_{i=1}^{2n}c_{i,0}r^i+\sum_{j=1}^{2n}c_{0,j}s^j+\sum_{\substack{1\leq i+j\leq 2n-1\\ i,j\neq 0}}c_{i,j}r^is^j$$

Therefore, taking into account that ν is a probability measure with moments or order i given by

$$m_i=\int_R r^i\,d\nu(r)$$

it holds:

$$\tilde{I}(\nu)=c_{0,0}+2\sum_{i=1}^{2n}c_{i,0}m_i+\sum_{\substack{1\leq i+j\leq 2n-1\\ i,j\neq 0}}c_{i,j}m_im_j.$$

Also, we have to notice that, from the definition of $\tilde{X}_\gamma$, the first moment is fixed, that is, $m_1 = \gamma$ and therefore, we can consider the vector of moments $\mathbf{m} = (m_2, \ldots, m_{2n})$ and define

$$\bar{W}(\mathbf{m}) = L(\mathbf{m}) + Q(\mathbf{m}),$$

where

$$L(\mathbf{m}) = c_{0,0} + 2c_{1,0}\gamma + 2\sum_{i=2}^{2n} c_{i0}m_i + c_{1,1}\gamma^2 + 2\sum_{i=2}^{2n-1} c_{i,1}\gamma m_i$$

is a linear form, and

$$Q(\mathbf{m}) = \sum_{\substack{2\le i+j\le 2n-1\\ i,j\neq 0,1}} c_{i,j}m_i m_j = \mathbf{m}^T C\mathbf{m}$$

is a quadratic form with matrix

$$C = \begin{pmatrix} c_{22} & \cdots & c_{2,2n-2} & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ c_{2n-2,2} & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 \end{pmatrix}$$

Therefore, the problem $(\tilde{P})$ is equivalent to the problem

$$\min_{\mathbf{m}\in\Omega} W(\mathbf{m}) \qquad (PM)$$

where Ω corresponds to the set of vectors in $\mathbb{R}^{2n-1}$ which are the moments of probability measures in $\tilde{X}_\gamma$. Due to the equivalence between both problems, we can assert that (PM) has a solution. Also a simple computation shows that $\nabla\bar{W} \neq 0$, because of the coercivity of W which implies $c_{2n,0} > 0$. Notice that, as consequence, the minimum in Ω has to attain at the boundary.