Coarse-convex-compactification approach to numerical solution of nonconvex variational problems

René Meziat[1] and Tomáš Roubíček[2,3]

[1] Departamento de Matemáticas, Universidad de los Andes
Carrera 1 No 18A-70, Bogotá, Colombia
[2] Mathematical Institute, Charles University
Sokolovská 83, CZ-186 75 Praha 8, Czech Republic
[3] Institute of Information Theory and Automation, Academy of Sciences
Pod vodárenskou věží 4, CZ-182 08 Praha 8, Czech Republic.

Abstract. In non-convex optimization problems, in particular in non-convex variational problems, there usually does not exist any classical solution but only generalized solutions which involve generalized Young functionals (e.g. Young measures or only some moments of them). In this paper, after reviewing briefly the relaxation theory for such problems, an approximation scheme based on coarse compactifications by only a finite number of moments and a finite-element approximation in the functional space of the problem is proposed and analyzed. Special attention is paid to problems involving polynomial nonlinearities, which leads to a relaxed formulation into a convex program based on linear matrix inequalities constraints with semidefinite programming structure. Finally, calculations of an illustrative 2D "broken-extremal" example are presented.

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1 Introduction

Nonconvex optimization problems often lack any solution because of fast oscillations of minimizing sequences that eventually break lower semicontinuity with respect to a weak convergence, cf. [69] and references therein for a survey on scalar variational problems which will be the first concern in this paper. Therefore, a relaxation is urgent to solve such problems in a suitably generalized sense. The most general way of relaxation is certainly a suitable continuous extension, using also a suitable linear-space structure not necessarily completely coherent with the linear structure occurring in the formulation of the original problem. Thus extended, (this is called a *relaxation*), nonconvex problems then may get a convex structure even if the original problem does not have any. For a large class of problems, a so-called *generalized Young functionals* (representing a generalization of conventional Young measures, cf. e.g. [65]) represent a suitable tool.

The relaxed problems can be discretized by a theory of convex approximation of the set of the generalized Young functionals developed recently in [63, 64, 65], see also [38, 55, 56] or a survey paper [67]. Thus the relaxed problem can directly be implemented on computer, without approximating the original non-relaxed problem; cf. [11, 26, 27, 36, 49, 50, 53, 54, 65, 68] for this approach. If the (additively coupled, cf. e.g. ($\mathfrak{P}$) below) problem is linear in a lower-order term (i.e. $\varphi_0(x,\cdot)$ in ($\mathfrak{P}$) is linear), such approach leads to a linear-programming problem and was shown very efficient in [3]. In the quadratic case, it naturally leads to a quadratic-programming problem, which is a considerably less efficient but still possible approach if the dimensionality is not too high, cf. [11, 36, 38, 68]. For non-quadratic case, one can still consider various iterative schemes, see Remark 3.9 below.

All the above mentioned references use conventional Young measures and treat them numerically in various more or less sophisticated ways. However, if the particular problem involves only a finite number of nonlinearities, it suffices to consider only moments of these Young measures with respect to these nonlinearities. This is the general idea of *coarse convex compatifications* as thoroughly exposed in [65]. In general, it is not easy to characterize explicitly such convex compactifications however. The goal of this paper is to exploit this alternative coarse-compactification approach in a particular case where the involved nonlinearities are *polynomials*. We show the success of this approach on a concrete problem of scalar multidimensional variational calculus with an *additively coupled* integral functional:

$$(\mathfrak{P}) \qquad \left\{ \begin{array}{ll} \text{Minimize} & \Phi(u) := \displaystyle\int_\Omega \varphi_1(x,\nabla u(x)) + \varphi_0(x,u(x))\,\mathrm{d}x, \\ \text{subject to} & u \in W^{1,p}(\Omega), \quad u|_{\partial\Omega} = u_{\mathrm{D}}, \end{array} \right.$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with Lipschitz boundary $\partial\Omega$ and the boundary condition $u_{\mathrm{D}} \in W^{1-1/p,p}(\partial\Omega)$ given; for more general problems see Remark 3.13 below. This paper formalizes ideas introduced in [39, 40, 41, 44, 45, 46] where authors use projections of Young measures onto finite dimensional convex bodies, in order to explicitly calculate the generalized solution of non-convex variational problems in Young measures.

The outline and the main contributions of this paper read as follows: In Section 2 we introduce the relaxation theory for the nonconvex variational problems ($\mathfrak{P}$) and an approximation by finite-element method. Section 3 is devoted to problems with polynomial nonlinearities and we also present there an explicit characterization of generalized Young functionals. In Section 4 we report on the performance of our algorithm applied to a benchmark model problem.

2 Relaxation by convex compactifications and its approximations

In this section we define the employed relaxation of ($\mathfrak{P}$) which is a continuous extension of ($\mathfrak{P}$) in terms of generalized Young functionals as suggested in [65, Chap.5]. We briefly state a construction of suitable envelopes of the Lebesgue space involved in ($\mathfrak{P}$), formulate the relaxed problem ($\mathfrak{RP}$) and the main results concerning the connections between ($\mathfrak{P}$) and ($\mathfrak{RP}$), as well as its approximation.

2.1 Convex local compactifications of L^p-spaces

Following [65, 66], we will briefly present a fairly universal construction of locally compact envelopes of the Lebesgue L^p-spaces that are also convex in a natural linear space and allow for a continuous and affine extension of Nemytskiĭ mappings. We assumed $\Omega \subset \mathbb{R}^n$ a bounded Lipschitz domain (here in Sect. 2.1, in fact, the Lipschitz property is not needed), and let us consider the Lebesgue space $L^p(\Omega;\mathbb{R}^m) = \{u : \Omega \to \mathbb{R}^m \text{ measurable; } \int_\Omega |u(x)|^p \mathrm{d}x < +\infty\}$. We define a normed linear space

$$\begin{aligned}\mathrm{Car}^p(\Omega;\mathbb{R}^m) := \Big\{ & h : \Omega \times \mathbb{R}^m \to \mathbb{R};\,, \\ & h(\cdot,s) \text{ measurable, } h(x,\cdot) \text{ continuous,} \\ & \exists a \in L^1(\Omega),\ b \in \mathbb{R} : \quad |h(x,s)| \le a(x) + b|s|^p \Big\}\end{aligned} \tag{2.1}$$

of Carathéodory's "test integrands", and dote it with the norm

$$\big\|h\big\|_{\mathrm{Car}^p(\Omega;\mathbb{R}^m)} := \inf_{|h(x,s)| \le a(x)+b|s|^p} \big\|a\big\|_{L^1(\Omega)} + b. \tag{2.2}$$

The essential trick is to consider a sufficiently large (but preferably still separable) linear subspace $H \subset \mathrm{Car}^p(\Omega;\mathbb{R}^m)$, to define the embedding

$$i_H : L^p(\Omega;\mathbb{R}^m) \to H^* : u \mapsto \Big(h \mapsto \int_\Omega h(x,u(x))\mathrm{d}x \Big), \tag{2.3}$$

and eventually to put

$$Y^p_H(\Omega;\mathbb{R}^m) := \text{ the weak* closure of } i_H(L^p(\Omega;\mathbb{R}^m)). \tag{2.4}$$

One can show that $Y^p_H(\Omega;\mathbb{R}^m)$ is always convex in H^*. Assuming, rather for simplicity, that H contains at least one coercive integrand, i.e. $H \ni h_0$ with $h_0(x,s) \geq |s|^p$, then $Y^p_H(\Omega;\mathbb{R}^m)$ is a *convex locally compact hull* of $L^p(\Omega;\mathbb{R}^m)$ and $L^p(\Omega;\mathbb{R}^m)$ itself is embedded into it (norm,weak*)-continuously. Moreover, if H is rich enough (cf. [65, 66] for details), then this embeddeding i_H is even homeomorphical. If H is separable, then $Y^p_H(\Omega;\mathbb{R}^m)$ is locally sequentially compact. Thus $Y^p_H(\Omega;\mathbb{R}^m)$ may be considered as indeed a very natural envelope of $L^p(\Omega;\mathbb{R}^m)$.

Moreover, let us define $h\bullet\eta$ as a Borel measure on $\bar\Omega$ by $\int_{\bar\Omega} g(x)[h\bullet\eta](\mathrm{d}x) = \langle h\bullet\eta, g\rangle = \langle \eta, gh\rangle$ where $[gh](x,s) = g(x)h(x,s)$ and $g \in C(\bar\Omega)$, where $\bar\Omega$ denotes the closure of Ω. Here we need that H is so-called $C(\bar\Omega)$-*invariant* in the sense that $gh \in H$ whenever $g \in C(\bar\Omega)$ and $h \in H$.

Further, we say that $\eta \in Y^p_H(\Omega;\mathbb{R}^m)$ is p-*nonconcentrating* if there is a sequence $\{u_k\}_{k\in\mathbb{N}}$ such that $\eta = \text{w*-}\lim_{k\to\infty} i_H(u_k)$ and $\{|u_k|^p;\ k \in \mathbb{N}\}$ is weakly relatively compact in $L^1(\Omega)$. Let us denote the set of all such η's by $\mathring{Y}^p_H(\Omega;\mathbb{R}^m)$.

If H is separable, any $\eta \in \mathring{Y}^p_H(\Omega;\mathbb{R}^m)$ has a L^p-*Young measure representation* ν in the sense that there is a weakly* measurable mapping $x \mapsto \nu_x$, ν_x a probablity measure on $\mathbb{R}^m$, such that $x \mapsto |s|^p\nu_x(\mathrm{d}s)$ belongs to $L^1(\Omega)$ and

$$\forall h \in H: \qquad \langle \eta, h\rangle = \int_\Omega\int_{\mathbb{R}^m} h(x,s)\,\nu_x(\mathrm{d}s)\mathrm{d}x, \tag{2.5}$$

see [65, Proposition 3.4.15]. It holds that $[\eta\bullet h](x) = \int_{\mathbb{R}^m} h(x,s)\,\nu_x(\mathrm{d}s)$ for a.a. $x \in \Omega$.

2.2 Relaxation of $(\mathfrak{P})$

We use the construction from Section 2.1 for $m = n$. We will assume that $\varphi_1 : \Omega\times\mathbb{R}^n \to \mathbb{R}$ and $\varphi_0 : \Omega\times\mathbb{R} \to \mathbb{R}$ are Carathéodory functions satisfying, for almost
ass1 all $x \in \Omega$, all $s \in \mathbb{R}^n$, and all $u \in \mathbb{R}$,

$$c_1|s|^p \leq \varphi_1(x,s) \leq c_2(1+|s|^p), \tag{2.6a}$$ ass1a

$$|\varphi_0(x,u)| \leq a(x) + c_3|u|^q, \tag{2.6b}$$ ass1b

where $p > 1$, $c_1, c_2, c_3 > 0$, $a \in L^1(\Omega)$, and $1 < q < pn/(n-p)$ if $p < n$ and $1 < q < \infty$ if $p \geq n$. Then we will consider the already announced relaxed problem in the form:

$$(\mathfrak{RP})\qquad \left\{\begin{array}{ll} \text{Minimize} & \bar\Phi(u,\eta) := \displaystyle\int_\Omega \big[\varphi_1\bullet\eta\big](x) + \varphi_0(x,u(x))\mathrm{d}x, \\ \text{subject to} & [\mathrm{Id}\bullet\eta](x) = \nabla u(x) \quad \text{for a.a. } x\in\Omega, \\ & u \in W^{1,p}(\Omega),\quad \eta \in Y^p_H(\Omega;\mathbb{R}^n),\quad u|_{\partial\Omega} = u_{\mathrm{D}}, \end{array}\right.$$

where $\mathrm{Id}(x,s) := s$; here we have to assume $\mathrm{Id} \in H^n$. The following assertion, which shows that $(\mathfrak{RP})$ is indeed a proper relaxation of $(\mathfrak{P})$, is based on results by Kinderlehrer and Pedregal [23, 57], therein we have:

thm1 **Proposition 2.1** (See [65, Propositions 5.2.1 and 5.2.6].) *Let (2.6) hold, $p > 1$, H be $C(\bar\Omega)$-invariant, $\varphi_1 \in H$, and $\mathrm{Id} \in H^n$. Then:*

(i) $(\mathfrak{RP})$ *admits a solution.*

(ii) $\inf(\mathfrak{P}) = \min(\mathfrak{RP})$.

(iii) *The embedding $e_H : W^{1,p}(\Omega) \to W^{1,p}(\Omega) \times Y^p_H(\Omega;\mathbb{R}^n) : v \mapsto (v, i_H(\nabla v))$ of any minimizing sequence for $(\mathfrak{P})$ has a weakly convergent subsequence whose (weak×weak*) limit is a solution to $(\mathfrak{RP})$.*

(iv) *Each solution to $(\mathfrak{RP})$ is p-nonconcentrating and is the (weak×weak*) limit of some minimizing sequence for $(\mathfrak{P})$ embedded via e_H.*

The consequence of Proposition 2.1(iv) is that we can replace $Y^p_H(\Omega;\mathbb{R}^n)$ by $\mathring{Y}^p_H(\Omega;\mathbb{R}^n)$ with an equal effect.

2.3 Finite-element discretization in space

sect-FE

As to the discretisation of Ω, we suppose that Ω is a polyhedral and we also consider, for a discretization mesh parameter $d > 0$ (ranging a countable set having 0 as its accumulation point), a triangulation $\mathcal{T}_d$ of Ω composed from simplexes $S \in \mathcal{T}_d$ such that $\max_{S\in\mathcal{T}_d} \mathrm{diam}(S) \le d$ and $\mathcal{T}_{d_1} \subset \mathcal{T}_{d_2}$ for $d_1 \ge d_2 > 0$, i.e. we consider nested triangulations refining everywhere on Ω when $d \searrow 0$. Then we define P_d by

$$[P_d h](x,s) := \frac{1}{\mathrm{meas}_n(S)} \int_S h(\xi,s)\,\mathrm{d}\xi \qquad \text{if } x \in S \in \mathcal{T}_d. \tag{2.7}$$

Requiring $P_d : H \to H$, we must consider such H which contains also discontinuous element-wise constant integrands. As we consider a sequence of triangulations $\mathcal{T}_d$, it is still possible to take H separable in the norm (2.2). In this norm, one can see that $\|P_d h\|_{\mathrm{Car}^p(\Omega;\mathbb{R}^m)} \le \|h\|_{\mathrm{Car}^p(\Omega;\mathbb{R}^m)}$ and $P_d \circ P_d = P_d$, so that $P_d : H \to H$ is a continuous projector. By [65, Proposition 3.5.9], it holds that $P_d^* Y^p_H(\Omega;\mathbb{R}^n) \subset Y^p_H(\Omega;\mathbb{R}^n)$. By [65, Proposition 3.5.2(iv)], we have $\bigcup_{d>0} P_d^* Y^p_H(\Omega;\mathbb{R}^n)$ weakly* dense in $Y^p_H(\Omega;\mathbb{R}^n)$. This suggests to approximate $(\mathfrak{RP})$ by restricting it on a convex subset $P_d^* Y^p_H(\Omega;\mathbb{R}^n)$ instead of $Y^p_H(\Omega;\mathbb{R}^n)$. Since any $\eta \in P_d^* Y^p_H(\Omega;\mathbb{R}^n)$ is element-wise constant, holding the constraint $\mathrm{Id}\bullet\eta = \nabla u$, the underlying u will then be automatically element-wise affine. By this way we come to the following *approximate relaxed problem*:

$$(\mathfrak{RP}_d) \qquad \left\{ \begin{array}{ll} \text{Minimize} & \bar\Phi(u,\eta) := \displaystyle\int_\Omega \big[\varphi_1\bullet\eta\big](x) + \varphi_0(x,u(x))\,\mathrm{d}x, \\ \text{subject to} & [\mathrm{Id}\bullet\eta](x) = \nabla u(x) \quad \text{for a.a. } x \in \Omega, \\ & u \in W^{1,p}(\Omega) \text{ element-wise affine on } \mathcal{T}_d, \\ & \eta \in P_d^* Y^p_H(\Omega;\mathbb{R}^n), \quad u|_{\partial\Omega} = u_\mathrm{D}. \end{array} \right.$$

Let us denote

$$G_0 := \bigcup_{d>0} \big\{ g \in L^\infty(\Omega);\quad \forall S \in \mathcal{T}_d : \quad g|_S \in C(\bar{S}) \big\} \tag{2.8}$$

where $g|_S \in C(\bar{S})$ means existence of a continuous extension on $\bar{S}$ of the restriction $g|_S$.

prop-FE **Proposition 2.2** (See [65, Proposition 5.5.1].) *Let H be separable, G-invariant, and satisfy $G \otimes V \subset H \subset \mathrm{cl}(G \otimes V)$ for some linear space $G \subset L^\infty(\Omega)$ containing G_0 and for some linear space V of continuous functions on $\mathbb{R}^n$ of at most p-growth, where the closure "*cl*" refers to the norm (2.2) and "$\otimes$" means the usual tensorial products, i.e. for functions $[g \otimes v](x,s) = g(x)v(s)$ and for spaces $G \otimes V$ is the linear span of $\{g \otimes v;\ g \in G,\ v \in V\}$. Then:*

(i) *A solution (u_d, η_d) to $(\mathfrak{RP}_d)$ always exists.*

(ii) *Moreover, $\lim_{d\to 0} \min(\mathfrak{RP}_d) = \min(\mathfrak{RP})$ and there always exists a subsequence of $d_i \to 0$ such that (u_{d_i}, η_{d_i}) (weak$\times$weak*)-converges in $W^{1,p}(\Omega) \times H^*$. Moreover, the limit of any such a subsequence solves $(\mathfrak{RP})$.*

Remark 2.3 As P_d is a projector, it holds that

$$\begin{aligned}\int_\Omega \varphi_1 \bullet \eta_d \,\mathrm{d}x &= \langle \eta_d, \varphi_1 \rangle = \langle P_d^* \eta_d, \varphi_1 \rangle = \langle P_d^* P_d^* \eta, \varphi_1 \rangle \\ &= \langle P_d^* \eta, P_d \varphi_1 \rangle = \langle \eta_d, P_d \varphi_1 \rangle = \int_\Omega (P_d \varphi_1) \bullet \eta_d \,\mathrm{d}x \end{aligned} \tag{2.9}$$

for any $\eta_d \in P_d^* Y_H^p(\Omega;\mathbb{R}^n)$, i.e. $\eta_d = P_d^* \eta$ for some $\eta \in Y_H^p(\Omega;\mathbb{R}^n)$, and therefore we can equally consider φ_1 in $(\mathfrak{RP}_d)$ replaced by its element-wise constant interpolant $P_d\varphi_1$. Also, by [65, Proposition 5.5.1(ii)], $P_d^* Y_H^p(\Omega;\mathbb{R}^n)$ in $(\mathfrak{RP}_d)$ can be replaced by $P_d^* \mathring{Y}_H^p(\Omega;\mathbb{R}^n)$ with an equal effect.

In the following remark, we focus on the Weierstrass Maximum Principle as a necessary condition for general variational problems.

WMP **Remark 2.4** Fixing $d > 0$, the *neccessary optimality conditions* for $(\mathfrak{RP}_d)$ which any solution (u_d, η_d) to $(\mathfrak{RP}_d)$ must satisfy are the existence of a vector field $\lambda_d \in L^\infty(\Omega;\mathbb{R}^n)$ element-wise constant satisfying, roughly speaking, one *half of the Euler-Lagrange equation* $\mathrm{div}\lambda_d = [\varphi_0]'_u(x, u_d)$ after discretized by finite elements, i.e.

$$\forall z \in W^{1,\infty}(\Omega) \text{ element-wise affine on } \mathcal{T}_d : \quad \int_\Omega \lambda_d \cdot \nabla z + \frac{\partial \varphi_0(x, u_d)}{\partial u} z \,\mathrm{d}x = 0, \tag{2.10}$$

and the *Weierstrass maximum principle* in the sense

$$\big[\lambda_d \otimes \mathrm{Id} - P_d \varphi_1\big] \bullet \eta_d = \max_{s \in \mathbb{R}^n} \Big(\lambda_d(x) \cdot s - [P_d \varphi_1](x,s) \Big) \qquad \text{for a.a. } x \in \Omega, \tag{2.11}$$

see [65, Sect.5.3] or [11]. The vector field λ_d is, in fact, the Lagrange multiplier to the constraint $\mathrm{Id}\bullet\eta = \nabla u$ in $(\mathfrak{RP}_d)$. If $\varphi_0(x,\cdot)$ is convex, then these optimality conditions are also sufficient. Indeed, formula (2.11) suggests the use of a relaxation in probability measures of the global optimization problem: $\max_{s\in\mathbb{R}^n}(\lambda_d(x)\cdot s - [P_d\varphi_1](x,s))$. This is just the approach presented in [28, 42, 43] for global optimization of polynomials.

3 Polynomial nonlinearities – method of moments

sect-poly

The definition (2.4) is very implicit. To implement the discretized relaxed problem on a computer, in view of Proposition 2.1(iv), we definitely need an explicit characterization of elements from $Y_H^p(\Omega;\mathbb{R}^n)$, or at least from $P_d\mathring{Y}_H^p(\Omega;\mathbb{R}^n)$, and possibly a further discretization of $P_d^*Y_H^p(\Omega;\mathbb{R}^n)$ if it is still infinite-dimensional, cf. [11, 38, 64, 65, 67]. Now we will focus on the case where all integrands from H are polynomials with order at most $2k$, $k\in\mathbb{N}$; i.e. $h(x,\cdot)\in\Pi_{2k}(\mathbb{R}^n)$ where $\Pi_{2k}(\mathbb{R}^n)$ is the family of all n-dimensional polynomials with degree $2k$ at the most. The decisive advantage of such choice is that P_dH is finite-dimensional and then $P_d^*Y_H^p(\Omega;\mathbb{R}^n)$ is automatically homeomorphic to a convex subset of a finite-dimensional Euclidean space; notice the constraints $m_{S,0,\cdots,0}=1$ for $S\in\mathcal{T}_d$ in $(\mathfrak{MP}_{d,k,\kappa})$ below. Therefore, in this case, no further discretization of $P_d^*Y_H^p(\Omega;\mathbb{R}^n)$ is needed.

We assume $k\in\mathbb{N}$ given and choose $p=2k$. Further choose

$$H = H_k := \sum_{l=1}^{2k-1} L^{p/(p-l)}(\Omega)\otimes\Pi_l(\mathbb{R}^n) + G_0\otimes\Pi_{2k}(\mathbb{R}^n) \tag{3.1}$$

where G_0 is from (2.8). Note that H_k is a linear subspace of $\mathrm{Car}^p(\Omega;\mathbb{R}^n)$ and, by the arguments presented in [66], it can be proved that it is separable. Hence it satisfies the assumptions of Proposition 2.2 with $G=G_0$ and $V=\Pi_{2k}(\mathbb{R}^n)$.

In this special case, every η can be represented by its moments

$$m_\iota = \eta\bullet(1\otimes s_1^{\iota_1}\cdots s_n^{\iota_n}) \tag{3.2}$$

where $\iota = (\iota_1,\cdots,\iota_n)$ is the multi-index of non-negative integers such that $|\iota| := \iota_1+\cdots+\iota_n \le 2k$. Namely, for any $h\in H_k$ with H_k from (3.1), i.e. $h=\sum_{|\iota|\le 2k} g_\iota(x)s_1^{\iota_1}\cdots s_n^{\iota_n}$ with uniquely determined $g_\iota\in L^{p/(p-|\iota|)}(\Omega)$, it holds that

$$\langle\eta,h\rangle = \sum_{|\iota|\le 2k}\langle\eta, g_\iota\otimes s_1^{\iota_1}\cdots s_n^{\iota_n}\rangle = \sum_{|\iota|\le 2k}\int_\Omega g_\iota(x)m_\iota(x)\,\mathrm{d}x. \tag{3.3}$$

Further, denoting $m=(m_\iota)_{|\iota|\le 2k}$, we define the matrix $\mathbb{H}_k(m)$ as

$$\mathbb{H}_k(m) := \big(m_{\iota_1+\iota_1',\cdots,\iota_n+\iota_n'}\big)_{0\le\iota_1+\iota_1'\le k,\cdots,0\le\iota_n+\iota_n'\le k} \tag{3.4}$$

and following current literature on global optimization of polynomials [28, 29, 30, 31, 32, 33, 34, 35, 58, 59, 60, 61] we define the *localizing* matrix $\mathbb{L}_k(m)$ as

$$\left(\varrho_d^2 m_{\iota_1+\iota_1',\cdots,\iota_n+\iota_n'} - m_{\iota_1+\iota_1'+2,\cdots,\iota_n+\iota_n'} - \cdots - m_{\iota_1+\iota_1',\cdots,\iota_n+\iota_n'+2}\right)_{0\leq\iota_1+\iota_1'\leq k-1,\cdots,0\leq\iota_n+\iota_n'\leq k-1} \tag{3.5}$$

It is well known that a probability measure μ on $\mathbb{R}^n$ induces a moments sequence $m_\iota = \int_{\mathbb{R}^n} s_1^{\iota_1}\cdots s_n^{\iota_n}\mathrm{d}\mu(\mathrm{d}s)$ which always makes the matrix $\mathbb{H}_k(m)$ positive semidefinite. The converse, i.e. existence of a probability measure μ inducing a prescribed sequence $(m_\iota)_{|\iota|\leq 2k}$ with $\mathbb{H}_k(m)\succeq 0$ and $m_0(x)=1$ as its moments, is unfortunately not true even in $n=1$ or even when $\mathbb{H}_k(m)\succ 0$ and $n>1$. The role of the localizing matrix $\mathbb{L}_k(m)$ is revealed when we focus on the family of measures supported on the n-dimensional ball $B_{\varrho_d} := \{s\in\mathbb{R}^n;\ s_1^2+\cdots+s_n^2\leq\varrho_d^2\}$. Thus, a probability measure μ on B_{ϱ_d} induces a moments sequence $m_\iota = \int_{B_{\varrho_d}} s_1^{\iota_1}\cdots s_n^{\iota_n}\mathrm{d}\mu(\mathrm{d}s)$ which makes the localizing matrix $\mathbb{L}_k(m)$ positive semidefinite. Even considering the localizing matrix $\mathbb{L}_k(m)$, the converse statement is no longer true again, however something useful can be done by applying recent characterizations of positive polynomials on compact semialgebraic sets like the ball B_{ϱ_d}. We will back on this issue below when we face the multidimensional case. See [28, 29, 34, 35, 58, 59, 60, 61].

3.1 The one-dimensional case

In the one-dimensional case, the matrix $\mathbb{H}_k(m) = (m_{\iota+\iota'})_{0\leq\iota+\iota'\leq k}$ takes the form of a *Hankel* matrix $[m_{\iota+\iota'}]_{\iota,\iota'=1}^k$. This one-dimensional case is particularly simple because the closure of the cone of moments of positive measures in the real line, i.e.

$$M = \{m\in\mathbb{R}^{2k+1} : m = \int_{\mathbb{R}}(1,t,\ldots,t^{2k})d\mu(t) \quad \text{for a positive measure } \mu \text{ in } \mathbb{R}\} \tag{3.6}$$

is precisely the cone of vectors $m\in\mathbb{R}^{2k+1}$ which make $\mathbb{H}_k(m)$ positive semidefinite. Although not every vector $m\in\mathbb{R}^{2k+1}$ satisfying this condition is a vector of moments, in the one dimensional case the coercivity of φ avoids any difficulty. The following lemma from [39, 40, 41] clarifies this point.

Lemma 3.1 *Let $\varphi(t)=\sum_{\iota=0}^{2k}c_\iota t^\iota$ be a one dimensional, coercive polynomial (i.e. $c_{2k}>0$). Therefore, any solution m^* of the semidefinite program:*

$$(\mathfrak{SDP}) \qquad \begin{cases} \textit{Minimize} & c\cdot m := \displaystyle\sum_{\iota=0}^{2k} c_\iota m_\iota \\ \textit{subject to} & \mathbb{H}_k(m)\succeq 0 \\ & \textit{with}\ \ m_0=1\ \textit{and}\ \ m_1=a, \end{cases}$$

is composed of the algebraic moments of a measure μ^* *satisfying the following abstract optimization problem defined in measures:*

$$(\mathfrak{AOP}) \qquad \begin{cases} \textit{Minimize} & \langle \varphi, \mu \rangle := \int_{\mathbb{R}} \varphi(t) d\mu(t) \\ \textit{subject to} & \int_{\mathbb{R}} t d\mu(t) = a \\ & \mu \in \mathcal{P}(\mathbb{R}) \end{cases}$$

where $\mathcal{P}(\mathbb{R})$ *stands for the family of all probability measures supported in the real line* $\mathbb{R}$*. The converse is also true, i.e. when* μ^* *solves* $(\mathfrak{AOP})$ *its algebraic moments solve* $(\mathfrak{SDP})$*.*

On one hand, this fact certainly allows us to determine exact relaxations in the one dimensional case. On the other hand, it also has an important geometrical meaning in convex analysis. Since the polynomial φ is coercive in $\mathbb{R}$, every point on the graph of its convex envelope φ_c can be expressed as a convex combination of points on the graph of φ itself. By applying classical Caratheodory's theorem in convex analysis we obtain the following formula:

$$(a, \varphi_c(a)) = \lambda_1(a_1, \varphi(a_1)) + \lambda_2(a_2, \varphi(a_2)) \tag{3.7}$$ convexa

where the coefficients λ_i represent a convex combination. It is remarkable that every optimal measure for $(\mathfrak{AOP})$ comes from the geometrical representation in (3.7). Thus, a probability measure $\overline{\mu}$ satisfies $(\mathfrak{AOP})$ if and only if it satisfies the equation:

$$(a, \varphi_c(a)) = \int_{\mathbb{R}} (t, \varphi(t)) d\overline{\mu}(t) \tag{3.8}$$

see [39, 40, 41, 42, 46]. From this observation we can see that

$$\overline{\mu} = \lambda_1 \delta_{a_1} + \lambda_2 \delta_{a_2} \tag{3.9}$$

is a solution of $(\mathfrak{AOP})$, where the coefficients λ_i and the points a_i, all together, satisfy (3.7). Thus, we can use a set of optimal values m^* from the semidefinite program $(\mathfrak{SDP})$ to determine the support and the weights of an optimal measures $\overline{\mu}$ solving $(\mathfrak{AOP})$. These facts can be used to prove the following result, later they will also be useful when applied into the multidimensional setting.

Proposition 3.2 *Assume that* $\varphi_1(x,t) = \sum_{\iota=0}^{2k} g_\iota(x) t^\iota$ *is a coercive, one dimensional polynomial in* t*, for almost every* $x \in [0,1]$ *then*

$$(\mathfrak{OCP}) \qquad \begin{cases} \textit{Minimize} & \Phi(m,u) = \int_0^1 \{\sum_{\iota=0}^{2k} g_\iota(x) m_\iota(x) + \varphi_0(u,x)\} dx \\ \textit{subject to} & m_0(x) = 1, \quad u'(x) = m_1(x) \\ & \mathbb{H}_k(m(x)) \succeq 0 \textit{ for every } x \in [0,1] \\ & u(0) = u_D(0), \quad u(1) = u_D(1) \end{cases}$$

is an exact relaxation of $(\mathfrak{RP})$ *in the one dimensional case, which has at least one solution. Moreover, every solution* m^* *of* $(\mathfrak{OCP})$ *can be traced back to a particular optimal* η^* *for the corresponding one dimensional case in the formulation* $(\mathfrak{RP})$*, in the sense that* m^* *is the vector function of the algebraic moments of the Young measure* η^**. That is* $m^*_\iota = \eta^* \bullet (1 \otimes s^\iota)$ *for* $\iota = 0, \cdots, 2k$*.*

This result has been proved and exploited constructively in [6, 39, 40, 41, 42, 46]. Notice that coercivity of φ_1 implies a finite, rather simple, support of the optimal measures of $(\mathfrak{RP})$. We would like to remark here, that in the one-dimensional case the relaxation of $(\mathfrak{RP})$ takes the form of a convex optimal control problem as $(\mathfrak{OCP})$, which must have at least a minimizer under coercivity assumptions. See [13, 18, 47, 48, 51, 72].

An analogous assertion holds for piecewise constant η's from $P_d\mathring{Y}^p_{H_k}(\Omega;\mathbb{R}^n)$. This fact suggests to formulate $(\mathfrak{RP}_d)$ in terms of moments. Let $\mathcal{T}_d$ be an equidistant partition with $d > 0$ a mesh size. Then the *approximate problem in terms of moments* looks as:

$$(\mathfrak{MP}_d)\qquad \begin{cases} \text{Minimize} & \hat{\Phi}(u,m) := \displaystyle\sum_{i=1}^{1/d}\sum_{\iota=0}^{2k} g_{i,\iota} m_{i,\iota} + \int_0^1 \varphi_0(x,u(x))\mathrm{d}x, \\ \text{subject to} & m_{i,0} = 1, \quad m_{i,1} = u'(x) \quad \text{for } x \in \big((i-1)d, id\big), \\ & \mathbb{H}_k(m_{i,0},\cdots,m_{i,2n}) \succeq 0 \quad \text{for all } \ i = 1,\cdots,1/d, \\ & u \in W^{1,p}(\Omega) \text{ element-wise affine on } \mathcal{T}_d, \\ & u(0) = u_D(0), \quad u(1) = u_D(1), \end{cases}$$

where the coefficients $g_{i,\iota}$ come from the expansion of the element-wise constant integrand $P_d\varphi_1$, i.e.

$$\big[P_d\varphi_1\big](x,s) = \sum_{\iota=0}^{2k} g_{i,\iota} s^\iota \quad \text{for } x \in \big((i-1)d, id\big). \tag{3.10}$$

Thus we turned the problem $(\mathfrak{RP}_d)$ into a *semidefinite programming* problem. Depending whether $\varphi_0(x,\cdot)$ is linear, convex quadratic, or more general, more or less efficient computer codes are available for solving it, e.g. the primal-dual interior point algorithm, generalized augmented-Lagrangian method [24], or a log-barrier method, respectively. For this approach, see [39, 40, 41, 46] where the last method has been applied.

From the analysis of the one dimensional case exposed above, the following equivalence clearly follows:

Proposition 3.3 *If* $n = 1$*, (2.6) requirements hold and* φ_1 *is a polynomial with degree* $2k$*, i.e. (3.10) holds, then the problem* $(\mathfrak{RP}_d)$ *with* H *from (3.1) is equivalent* *to* $(\mathfrak{MP}_d)$ *in the sense that:*

(i) $\min(\mathfrak{RP}_d) = \min(\mathfrak{MP}_d)$, *and*

(ii) (u^*, η^*) *solves* $(\mathfrak{RP}_d)$ *if and only if* (u^*, m^*) *solves* $(\mathfrak{MP}_d)$, *where optimal* η^* *is related to optimal* m^* *through (3.2), which means here* $m^*_\iota = \eta^* \bullet (1 \otimes s^\iota)$ *for* $\iota = 0, \cdots, 2k$.

It is worth to mention here, that we can use algebraic tools for constructing finite supported measures from a finite set of their moments, so we can use the optimal vectors m^*_d from $(\mathfrak{MP}_d)$ to estimate a η^*_d as a minimizer of $(\mathfrak{RP}_d)$. See [42, 43].

3.2 The multi-dimensional case

The Multidimensional Moment Problem is still an open problem in pure and applied mathematics see [1, 5, 12, 22, 25, 70]. Nonetheless, important progress has been made in recent years as algebraists have been able to characterize positive polynomials defined in compact semi-algebraic sets [14, 15, 16, 19, 21, 35, 71]. This result has been applied to global optimization of polynomials and non-convex situations in optimization theory, see [28, 29, 30, 31, 32, 33, 34, 35, 39, 40, 41, 42, 43, 46, 52, 58, 59, 60, 61]. We use this methodology here to transform $(\mathfrak{RP}_d)$ into a convex optimization problem in which the Young-measures ν_d are represented as moments-like vectorial functions m_d within a proper convex mathematical program. In particular we will follow J.B. Lasserres's approach on global optimization of polynomials given in [28, 29, 30, 31, 32, 33] to describe the convex envelope of n-dimensional coercive polynomials by semidefinite programming.

The case $n > 1$ is much more complicated because the characterization of elements η from $\mathring{Y}^p_H(\Omega; \mathbb{R}^n)$ only can be attained in a limit sense, provided that the supports of the parametrized measures in the Young measure representation of η lie in a compact semi-algebraic set. Thus, we assume that every parametrized measure ν_x is supported on the n-dimensional ball: $B_{\varrho_d} := \{s \in \mathbb{R}^n;\, s_1^2 + \cdots + s_n^2 \le \varrho_d^2\}$ where ϱ_d is chosen to convenience. Due to the uniform coercivity of φ_1, cf. (2.6a), the maximum on the right-hand side of (2.11) can be achieved inside a ball B_{ϱ_d} for ϱ_d sufficiently large. By proceeding in this way, we can apply recent results on the characterization of multidimensional moments without changing the original formulation of the problem.

It is enlightening for our approach to focus on the convex envelope of the integrand $\varphi_1(x, \nabla u(x))$ with respect to the gradient variables. Indeed, it has been observed by other authors that this kind of convexification allows us to obtain an exact convex relaxation of the original non-convex problem $(\mathfrak{P})$. Thus, when we consider the convex formulation:

$$
(\mathfrak{CP}) \qquad \begin{cases} \text{Minimize} & \Phi_c(u) := \displaystyle\int_\Omega C\varphi_1(x, \nabla u(x)) + \varphi_0(x, u(x))\ \mathrm{d}x, \\ \text{subject to} & u \in W^{1,p}(\Omega), \quad u|_{\partial\Omega} = u_{\mathrm{D}}, \end{cases}
$$

we obtain an exact relaxation of ($\mathfrak{P}$) see [17, 20, 57, 65]. After a proper discretization as described in Section 2, we must center our attention on the discretized convexified problem:

$$
(\mathfrak{CP}_d) \qquad \begin{cases} \text{Minimize} & \Phi_c(u) := \displaystyle\int_\Omega C\varphi_1(x, \nabla u(x)) + \varphi_0(x, u(x))\ \mathrm{d}x, \\ \text{subject to} & u \in W^{1,p}(\Omega),\ \text{element-wise affine on } \mathcal{T}_d, \quad u|_{\partial\Omega} = u_{\mathrm{D}}. \end{cases}
$$

We will prove next that ($\mathfrak{CP}_d$) is also equivalent to ($\mathfrak{RP}_d$) under proper coercivity assumptions.

lema **Proposition 3.4** (See [57].) *By assuming the coercivity requirements on φ_1 given in 2.6a, we have the following results:*

1. *Let u^* be a solution for ($\mathfrak{CP}$). The couple (u^*, η^*) with $\eta^* \in Y^p_H(\Omega; \mathbb{R}^n)$ is a solution of ($\mathfrak{RP}$) provided that $C\varphi_1(x, \nabla u^*(x)) = \big[\varphi_1 \bullet \eta^*\big](x)$ and $[\mathrm{Id} \bullet \eta^*](x) = \nabla u^*(x)$ for almost every $x \in \Omega$.*

2. *Reciprocally, if (u^*, η^*) solves ($\mathfrak{RP}$), the function u^* is a solution for ($\mathfrak{CP}$) and they solve the equations $C\varphi_1(x, \nabla u^*(x)) = \big[\varphi_1 \bullet \eta^*\big](x)$ and $[\mathrm{Id} \bullet \eta^*](x) = \nabla u^*(x)$ for almost every $x \in \Omega$.*

Dacorogna **Lemma 3.5** (See [17].) *Given a n-dimensional polynomial $\varphi : \mathbb{R}^n \to \mathbb{R}$ satisfying $c_1|s|^p \le \varphi(s) \le c_2(1 + |s|^p)$ for every $s \in \mathbb{R}^n$, with $p > 1$ and positive constants c_1 and c_2, we can determine its convex envelope at a fixed point $a \in \mathbb{R}^n$ as:*

$$
(\mathfrak{ECP}) \qquad C\varphi(a) = \begin{cases} \textit{Minimize} & \langle \varphi, \mu \rangle := \displaystyle\int_{\mathbb{R}^n} \varphi(s) d\mu(s) \\ \textit{subject to} & \int_{\mathbb{R}^n} s d\mu(s) = a \\ & \mu \in \mathcal{P}(\mathbb{R}^n) \end{cases}
$$

where $\mathcal{P}(\mathbb{R}^n)$ is the family of all Borel regular, probability measures supported in $\mathbb{R}^n$.

coment **Remark 3.6** At the boundary point $(a, C\varphi(a))$ there exists a supporting hiperplane for the convex set $Epi(C\varphi)$. Such hiperplane can be defined by a linear-afin function $L_a : \mathbb{R}^n \to \mathbb{R}$ which satisfies $L_a \le C\varphi \le \varphi$ and $L_a(a) = \varphi(a)$. Thus, we easily characterize the optimal measures μ^* for ($\mathfrak{ECP}$) as the set of probability measures supported in $\mathbb{F}_a = \{s \in \mathbb{R}^n : L_a(s) = \varphi(s)\}$ satisfying $a = \int_{\mathbb{R}^n} s d\mu^*(s)$. Hence, we find that a necessary condition for μ^* to be optimal in $(\mathfrak{CCP})$ is that μ^* be supported in $\{s \in \mathbb{R}^n : C\varphi(s) = \varphi(s)\}$. Reader should notice why this condition is not sufficient.

lema2 **Proposition 3.7** *By assuming the coercivity requirements on φ_1 given in 2.6a, the discrete problem $(\mathfrak{CP}_d)$ is equivalent to $(\mathfrak{RP}_d)$ in the following sense:*

1. *Let u_d^* be a solution for $(\mathfrak{CP}_d)$. The couple (u_d^*, η_d^*) with $\eta^* \in P_d^* Y_H^p(\Omega; \mathbb{R}^n)$ is a solution of $(\mathfrak{RP}_d)$ provided that $C\varphi_1(x, \nabla u_d^*(x)) = [\varphi_1 \bullet \eta_d^*](x)$ and $[\mathrm{Id} \bullet \eta_d^*](x) = \nabla u_d^*(x)$ for every $S \in \mathcal{T}_d$.*

2. *Reciprocally, if (u_d^*, η_d^*) solves $(\mathfrak{RP}_d)$, the function u_d^* is a solution for $(\mathfrak{CP}_d)$ and they solve the equations:*

$$C\varphi_1(x, \nabla u_d^*(x)) = [\varphi_1 \bullet \eta_d^*](x) \quad \text{and} \quad [\mathrm{Id} \bullet \eta_d^*](x) = \nabla u_d^*(x) \tag{3.11}$$

for every $S \in \mathcal{T}_d$. Notice that equations 3.11 are satisfied in an element wise way, being constant inside every triangle $S \in \mathcal{T}_d$.

Proof.

1. Let (u_d, η_d) be an admissible solution for the relaxed problem $(\mathfrak{RP}_d)$, then $[\mathrm{Id} \bullet \eta_d](x) = \nabla u_d(x)$ for every $S \in \mathcal{T}_d$. Since u_d^* is optimal for $(\mathfrak{CP}_d)$, we have:

$$\int_\Omega C\varphi_1(x, \nabla u_d(x)) + \varphi_0(x, u_d(x))\ \mathrm{d}x \geq \int_\Omega C\varphi_1(x, \nabla u_d^*(x)) + \varphi_0(x, u_d^*(x))\ \mathrm{d}x \tag{3.12}$$

By using Lemma 3.5, we can see that

$$\int_\Omega [\varphi_1 \bullet \eta_d](x) + \varphi_0(x, u_d(x))\ \mathrm{d}x \geq \int_\Omega C\varphi_1(x, \nabla u_d(x)) + \varphi_0(x, u_d(x))\ \mathrm{d}x \tag{3.13}$$

and finally we have

$$\int_\Omega [\varphi_1 \bullet \eta_d](x) + \varphi_0(x, u_d(x))\ \mathrm{d}x \geq \int_\Omega [\varphi_1 \bullet \eta_d^*](x) + \varphi_0(x, u_d^*(x))\ \mathrm{d}x \tag{3.14}$$

because of the assumptions on u^* and η^*. Thus, we have shown that (u_d^*, η_d^*) is optimal for the problem $(\mathfrak{RP}_d)$.

2. Let η_d^{**} be the Young measure induced by the convex envelope of φ_1 according to Lemma 3.5, i.e.

$$C\varphi_1(x, \nabla u_d^*(x)) = \left[\varphi_1 \bullet \eta_d^{**}\right](x) \quad \text{and} \quad [\mathrm{Id} \bullet \eta_d^{**}](x) = \nabla u_d^*(x) \tag{3.15}$$

for every $S \in \mathcal{T}_d$. As (u_d^*, η_d^{**}) is admissible and (u_d^*, η_d^*) is optimal for the problem $(\mathfrak{RP}_d)$, we have

$$\int_\Omega [\varphi_1 \bullet \eta_d^{**}](x) + \varphi_0(x, u_d^*(x)) \ \mathrm{d}x \geq \int_\Omega [\varphi_1 \bullet \eta_d^*](x) + \varphi_0(x, u_d^*(x)) \ \mathrm{d}x \tag{3.16}$$

then

$$\int_\Omega C\varphi_1(x, \nabla u_d^*(x))\mathrm{d}x = \int_\Omega [\varphi_1 \bullet \eta_d^{**}](x)\mathrm{d}x \geq \int_\Omega [\varphi_1 \bullet \eta_d^*](x)\mathrm{d}x \geq \int_\Omega C\varphi_1(x, \nabla u_d^*(x))\mathrm{d}x \tag{3.17}$$

Hence,

$$\int_\Omega [\varphi_1 \bullet \eta_d^*](x)\mathrm{d}x = \int_\Omega C\varphi_1(x, \nabla u_d^*(x))\mathrm{d}x. \tag{3.18}$$

By applying Lemma 3.5 again, we can claim that

$$C\varphi_1(x, \nabla u_d^*(x)) \leq [\varphi_1 \bullet \eta_d^*](x) \tag{3.19}$$

for every $S \in \mathcal{T}_d$. But the integrals in 3.18 can be expressed as a finite sum on the members of $\mathcal{T}_d$. Thus, we have:

$$\sum_{S \in \mathcal{T}_d} \int_S [\varphi_1 \bullet \eta_d^*](x)\mathrm{d}x = \sum_{S \in \mathcal{T}_d} \int_S C\varphi_1(x, \nabla u_d^*(x))\mathrm{d}x. \tag{3.20}$$

Therefore, we can conclude that:

$$[\varphi_1 \bullet \eta_d^*](x) = C\varphi_1(x, \nabla u_d^*(x)) = [\varphi_1 \bullet \eta_d^{**}](x) \tag{3.21}$$

for every $S \in \mathcal{T}_d$. In this way we can see that $\eta_d^{**} = \eta_d^*$. Herein that

$$C\varphi_1(x, \nabla u_d^*(x)) = \left[\varphi_1 \bullet \eta_d^*\right](x) \quad \text{and} \quad [\mathrm{Id} \bullet \eta_d^*](x) = \nabla u_d^*(x) \tag{3.22}$$

for every $S \in \mathcal{T}_d$. To see that u_d^* is optimal for the convexified problem $(\mathfrak{CP}_d)$, we take an admissible u_d for $(\mathfrak{CP}_d)$ and we define η_d as the Young measure induced by the convex envelope of φ_1 according to Lemma 3.5. Therefore, we have:

$$\begin{aligned} &\int_\Omega C\varphi_1(x, \nabla u_d(x)) + \varphi_0(x, u_d(x)) \ \mathrm{d}x \\ &= \int_\Omega [\varphi_1 \bullet \eta_d](x) + \varphi_0(x, u_d(x)) \ \mathrm{d}x \\ &\geq \int_\Omega [\varphi_1 \bullet \eta_d^*](x) + \varphi_0(x, u_d^*(x)) \ \mathrm{d}x \\ &= \int_\Omega C\varphi_1(x, \nabla u_d^*(x)) + \varphi_0(x, u_d^*(x)) \ \mathrm{d}x. \end{aligned} \tag{3.23}$$

In this way we conclude that u_d^* is optimal for $(\mathfrak{CP}_d)$.

$\blacksquare$

From Proposition 3.7, it is clear the role of the convex envelope of the polynomial φ_1 into the relaxed formulation $(\mathfrak{RP}_d)$. From Lemma 3.5, it is also clear that the convex envelope of a multidimensional polynomial φ_1 can be described by a particular optimization problem defined in probability measures. Thus, we can apply the moments technique to the relaxed formulation $(\mathfrak{RP}_d)$, taking into account that we must obtain at last the convex envelope of the polynomial φ_1 to be certain that we are in presence of an exact relaxation of the original problem. See [44]. The following result supports such approach.

$\boxed{\texttt{propo}}$ **Proposition 3.8** *Consider the convex, relaxed and discretized problem:*

$$(\mathfrak{MP}_{d,\kappa}) \qquad \left\{ \begin{array}{ll} \textit{Minimize} & \hat{\Phi}(u,m) := \displaystyle\sum_{S\in\mathcal{T}_d}\sum_{\iota=0}^{k} \phi_{S,\iota} m_{S,\iota} + \int_\Omega \varphi_0(x,u(x))\mathrm{d}x, \\ \textit{subject to} & m_{S,e_i} = \dfrac{\partial u}{\partial x_i} \quad \textit{on } S\in\mathcal{T}_d, \quad i=1,\cdots,n, \\ & \left.\begin{array}{l} m_{S,0,\cdots,0} = 1, \\ \mathbb{H}_\kappa\big(\{m_{S,\iota}\}_{|\iota|\le 2\kappa}\big) \succeq 0, \\ \mathbb{L}_\kappa\big(\{m_{S,\iota}\}_{|\iota|\le 2(\kappa-1)}\big) \succeq 0, \end{array}\right\} \ \textit{for all} \ \ S\in\mathcal{T}_d, \\ & u\in W^{1,p}(\Omega) \ \textit{element-wise affine of } \mathcal{T}_d, \quad u|_{\partial\Omega} = u_{\mathrm{D}}, \end{array}\right.$$

where $e_i = (0,\cdots,0,1,0,\cdots,0)\in\mathbb{R}^n$ is the vector with 1 on the i-th position and where, similarly as in $(\mathfrak{MP}_{d,k})$, the coefficients $\phi_{S,\iota}$ come from the expansion of the element-wise constant integrand $P_d\varphi_1$, i.e. $[P_d\varphi_1](x,s) = \sum_{\iota=0}^{k}\phi_{S,\iota}s_1^{\iota 1}\cdots s_n^{\iota n}$ for $x\in S\in\mathcal{T}_d$. We claim that:

1. *$(\mathfrak{MP}_{d,\kappa})$ has a solution for every $d>0$, k and κ with $\kappa\ge k$.*
2. *The solution of $(\mathfrak{MP}_{d,\kappa})$ provides a lower bound for $(\mathfrak{RP}_d)$.*
3. *$\min(\mathfrak{MP}_{d,\kappa}) \nearrow \inf(\mathfrak{RP}_d)$ when $\kappa\to\infty$.*
4. *When (u^*,m^*) solves $(\mathfrak{MP}_{d,\kappa})$ and there exists $\eta^*\in P_d^*Y_H^p(\Omega;\mathbb{R}^n)$ such that $m_\iota^* = \eta^*\bullet(1\otimes s_1^{\iota 1}\cdots s_n^{\iota n})$ for $|\iota| := \iota_1+\cdots+\iota_n \le 2k$, then η^* solves $(\mathfrak{RP}_d)$.*

Proof.

We use again some tools from convex optimization and convex analysis to understand the convex envelope of a coercive polynomial $\varphi:\mathbb{R}^n\to\mathbb{R}$ at the point $a=(a_1,\ldots,a_n)$. The coefficients of the polynomial $\varphi(x) = \sum_{2\le|\iota|\le k} c_{\iota_1,\ldots\iota_n} x_1^{\iota 1}\ldots x_n^{\iota n}$ allow us to define the following semidefinite program:

$$(\mathfrak{SDP}_\kappa) \qquad \begin{cases} \text{Minimize} & c \cdot m := \displaystyle\sum_{2 \le |\iota| \le k} c_{\iota_1,\dots\iota_n} m_{\iota_1,\dots\iota_n} \\ \text{subject to} & \mathbb{H}_\kappa(m) \succeq 0 \ \text{ and } \mathbb{L}_\kappa(m) \succeq 0 \\ & \text{with } m_{0,\dots,0} = 1 \text{ and } m_{e_1} = a_1, \dots, m_{e_n} = a_n. \end{cases}$$

Notice that $\kappa \ge k$, in this way $(\mathfrak{SDP}_\kappa)$ is a sequence of convex programs indexed by κ. Notice that we omit the linear part of φ as it does not affect the analysis of the convex envelope of the polynomial φ. Since $\mathbb{H}_\kappa(m) \succeq 0$ and $\mathbb{L}_\kappa(m) \succeq 0$ are necessary conditions for m to be a valid sequence of multidimensional moments of a multidimensional measure supported in the n-dimensional ball B_{ϱ_d}, the optimal value of $(\mathfrak{SDP}_\kappa)$ is a lower bound for the value $C\varphi(a)$. We have used here the right hand of $(\mathfrak{ECP})$. If every program $(\mathfrak{SDP}_\kappa)$ has a finite optimal value, then the optimal values of $(\mathfrak{SDP}_\kappa)$ define a nondecreasing sequence of lower bounds of $C\varphi(a)$. We will show that the optimal value of $(\mathfrak{SDP}_\kappa)$ converges to $C\varphi(a)$ when $\kappa \to \infty$ by following J.B. Lasserre's proposal for global optimization of polynomials stated in the seminal paper [28].

Let L_a be the linear-afin function defining the supporting hiperplane of the convex set $Epi(C\varphi) \subseteq \mathbb{R}^{n+1}$ at the point $(a, C\varphi(a))$. See Remark (3.6). Given $\varepsilon > 0$, we have $\varphi(x) - L_a(x) + \varepsilon > 0$ for every $x \in B_{\varrho_d}$. Since the ball B_{ϱ_d} is a semialgebraic compact set, we can express the positive polynomial $\varphi(x) - La(x) + \varepsilon$ in B_{ϱ_d} as:

$$(\mathfrak{QF}) \qquad \varphi(x) - L_a(x) + \varepsilon = \sum_{j=1}^{J} q_j^2(x) + (\varrho_d^2 - x_1^2 - \cdots - x_n^2) \sum_{j'=1}^{J'} q_{j'}^2$$

where q_j and $q_{j'}$ are n-dimensional polynomials whose degrees can not be determined in advance, i.e. they depend on ε. See [62]. If we take κ as the degree of the polynomial at the right side of $(\mathfrak{QF})$, then the quadratic representation of $(\mathfrak{QF})$ gives a feasible solution of the dual of the semidefinite program $(\mathfrak{SDP}_\kappa)$. See [28, 44].

The dual form of the semidefinite program $(\mathfrak{SDP}_\kappa)$ is

$$(\mathfrak{D}) \qquad \begin{cases} \text{Minimize} & -\gamma_{0,\dots,0} - \lambda_{0,\dots,0} - 2a_1\gamma_{e_1} - \cdots - 2a_n\gamma_{e_n} \\ \text{subject to} & \langle \Gamma, A_{\iota_1,\dots\iota_n} \rangle_\kappa + \langle \Lambda, \widetilde{A}_{\iota_1,\dots\iota_n} \rangle_{\kappa-1} = c_{\iota_1,\dots\iota_n} \\ & \text{for every } \iota \text{ satisfying } 2 \le |\iota| \le \kappa \quad \text{, with} \\ & \mathbb{H}_\kappa(\gamma) \succeq 0 \text{ and } \mathbb{L}_\kappa(\lambda) \succeq 0. \end{cases}$$

The primal semidefinite program $(\mathfrak{SDP}_\kappa)$ is strictly feasible as we can always find a set of moments m, induced by a continuously distributed probability measure in B_{ϱ_d} whose first marginal moments are the values $a_1, \dots, a_n$. See [14]. In this way the moments m define positive definite matrices $\mathbb{H}_\kappa(m)$ and $\mathbb{L}_\kappa(m)$. As usual in convex optimization, feasible solutions of the dual program $(\mathfrak{D})$ provide lower bounds for the primal program $(\mathfrak{SDP}_\kappa)$. So, as proposed in [28], we take a couple of dual variables Γ and Λ from the coefficients of the polynomials q_j and $q_{j'}$ in $(\mathfrak{QF})$. This is, $\Gamma = \sum_{j=1}^{J} q_j \cdot q_j^t$ and $\Lambda = \sum_{j'=1}^{J'} q_{j'} \cdot q_{j'}^t$ which implies that matrices $\mathbb{H}_\kappa(\gamma) \succeq 0$ and $\mathbb{L}_\kappa(\lambda)$ must be semidefinite positive. From the representation $(\mathfrak{QF})$ of the positive polynomial $\varphi(x) - L_a(x) + \varepsilon$ we can see that:

$$-\gamma_{0,\ldots,0} - \lambda_{0,\ldots,0} - 2a_1\gamma_{e_1} - \cdots - 2a_n\gamma_{e_n} = L_a(a) - \varepsilon.$$

Therefore, the dual variables Γ and Λ determine the following lower bound for $(\mathfrak{SDP}_\kappa)$:

$$-\gamma_{0,\ldots,0} - \lambda_{0,\ldots,0} - 2a_1\gamma_{e_1} - \cdots - 2a_n\gamma_{e_n} = C\varphi(a) - \varepsilon.$$

As every optimal value in $(\mathfrak{SDP}_\kappa)$ is a lower bound for $C\varphi(a)$, we have:

$$C\varphi(a) \geq \delta_\varepsilon^* \geq C\varphi(a) - \varepsilon$$

where δ_ε^* is the optimal value of $(\mathfrak{SDP}_\kappa)$, which is finite because the primal program $(\mathfrak{SDP}_\kappa)$ is strictly feasible.

The present analysis of the convex envelope of a n-dimensional polynomial proves items 1-4 of Proposition (3.8) when applied on the coercive polynomial φ_1 given in $(\mathfrak{RP}_d)$ provided that φ_0 is convex in u. For a good introduction to duality results of semidefinite programming we refer to [4]. ■

3.3 Remarks

Let us end this section with few remarks that outlines combination with various advanced numerical strategies and various generalizations.

rem-SLP **Remark 3.9** One can think about a linearization of the φ_0-term in the relaxed problem $(\mathfrak{RP}_d)$ and then, starting from some u^0, consider the iterative process based for $k = 1, 2, \ldots$ on the solution $(u_d^{(k)}, \eta_d^{(k)})$ to the problem

$$\begin{aligned}
&\text{Minimize} && \Psi(u_d^{(k-1)}; u, \eta) := \int_\Omega \big[\varphi_1 \bullet \eta\big](x) + [\varphi_0]_u'(x, u_d^{(k-1)})(u - u_d^{(k-1)})\mathrm{d}x,\\
&\text{subject to} && [\mathrm{Id} \bullet \eta](x) = \nabla u(x) \quad \text{for a.a. } x \in \Omega,\\
& && u \in W^{1,p}(\Omega) \text{ element-wise affine on } \mathcal{T}_d,\\
& && \eta \in P_d^* Y_H^p(\Omega; \mathbb{R}^n), \qquad u|_{\partial\Omega} = u_{\mathrm{D}}, \boxed{\texttt{aqui}}
\end{aligned}$$

where $u_d^{(k-1)}$ is known from the previous iteration. This SLP (=*sequential linear programming*) strategy was proposed in [3]. In some qualified cases (e.g. in the benchmark problem in Section 4 below), it converges by the Banach fixed-point argument, see [3] for details. Here it may open a possibility, after applying the methods of moments from Section 3, to use efficient semidefinite programming algorithms with linear cost functional even if $\varphi_0(x,\cdot)$ is nonlinear and nonconvex. For an SQP (=*sequential quadratic programming*) strategy, which allows for usage of still relatively efficient semidefinite programming algorithms with quadratic cost functional even if $\varphi_0(x,\cdot)$ is non-quadratic, see [37].

Remark 3.10 By solving relaxation ($\mathfrak{MP}_{d,\kappa}$) with increasing values of κ, we eventually attain an exact solution or at least a good approximation of the exact moments values. See [28]. By doing so, we obtain a set of parametrized multidimensional moments $m^*(S)$ for every $S \in \mathcal{T}_d$. With this optimal moments we can calculate the optimal Young measure on every $S \in \mathcal{T}_d$. This procedure exploits the marginal algebraic moments and the convex hull properties of the non-convex potential φ_1 theorem in convex analysis. See [44, 45] for further details of this implementation.

Remark 3.11 Some semidefinite programming algorithms yield the Lagrange multipliers $\lambda_{d,k,\kappa}$ to the first constraint in ($\mathfrak{RP}_{d,k,\kappa}$), i.e. $(m_{S,\iota})_{|\iota|=1} = \nabla u$. They can be used as a certain approximation of the multipliers λ_d in (2.10)–(2.11) and also the multipliers that can determine by Weierstrass' principle like (2.11) approximately the support of a Young measure still discretized additionally by restricting it on a convex combination of a finite number of Dirac measures supproted by a-priori selected points of $\mathbb{R}^n$. This is now called an active-set strategy algorithm, developed originally in [11] and latter used e.g. in [3, 27, 36, 37, 68]. In this way, we would get an upper estimate for $\min(\mathfrak{RP}_d)$ which would complete the previously obtained lower estimate $\min(\mathfrak{MP}_{d,k,\kappa})$.

Remark 3.12 Considering $\kappa \geq k$ and only the part of this solution, namely the moments $m_{S,\iota}$ with $|\iota| \leq 2k$, we can define $\eta_\kappa \in H^*$ by (3.3) where $m_\iota(x) = m_{S,\iota}$ for $x \in S \in \mathcal{T}_d$. Unfortunately, except special cases as in Sect. 4, such η_κ need not belong to $\mathring{Y}^p_{H_k}(\Omega;\mathbb{R}^n)$ even in a limit for $\kappa \to \infty$, contrary to the one-dimensional case.

non-add **Remark 3.13** In principle, in contrast to the additively coupled problem ($\mathfrak{P}$), we could consider more general problems involving the generally coupled functional $\Phi(u) := \int_\Omega \varphi(x, u(x), \nabla u(x))\mathrm{d}x$. Our results allow relatively easily for an extension to the case $\varphi(x,u,s) = \sum_{|\iota|\leq 2k} g_\iota(x,u(x)) s_1^{\iota_1}\cdots s_n^{\iota_n}$. Then the φ_0-term in ($\mathfrak{MP}_{d,k}$) and ($\mathfrak{MP}_{d,k,\kappa}$) would be out but the coefficients $\phi_{i,\iota}$ and $\phi_{S,\iota}$ would depend on u, which would turn ($\mathfrak{MP}_{d,k}$) and ($\mathfrak{MP}_{d,k,\kappa}$) into general nonconvex positive-semidefinite mathematical programs.

4 Illustrative two-dimensional example

-benchmark

We show effectiveness of the proposed approach on a benchmark problem used already in [3, 37], namely the so-called Tartar's broken-extremal example [54] modified for the two-dimensional case like in [8, Sect.8]. To be more specific, let us consider the square $\Omega := (0, K)^n$ with $n = 2$ with the side $K > 0$ and, for almost all $x \in \Omega$, all $s \in \mathbb{R}^2$, and all $u \in \mathbb{R}$,

$$\varphi_1(x, s) := |s - a|^2 |s + a|^2, \tag{4.1a}$$

F

$$\varphi_0(x, u) := \big(u - g(a \cdot x)\big)^2 \quad \text{with} \tag{4.1b}$$

G

$$g(\xi) := -\frac{3}{128}(\xi - \xi_{\rm b})^5 - \frac{1}{3}(\xi - \xi_{\rm b})^3, \tag{4.1c}$$

g

for $a = (\cos\phi, \sin\phi)$ with $\phi = \pi/6$ and for $\xi_b = 1/2$. Note that (4.1a) (when shifted by a constant) fits with (2.6a) for $p = 4$ and that $\varphi_1(x, \cdot) \in \Pi_4(\mathbb{R}^2)$, hence Section 3 applies for $k = 2 = n$; note that $\varphi_1(x, s)$ from (4.1a) is indeed a polynomial of the 4th order in terms of $s = (s_1, s_2)$:

$$\varphi_1(x, s_1, s_2) \;=\; s_1^4 + s_2^4 + 2s_1^2 s_2^2 - s_1^2 + s_2^2 - 2\sqrt{3}s_1 s_2 + 1. \tag{4.2}$$

exact-sln

Then according to (cf. [54]), the relaxed problem $(\mathfrak{RP})$ has the unique solution

$$u(x) = \begin{cases} g(a \cdot x) & \text{for } a \cdot x \in (0, \xi_{\rm b}), \\ \dfrac{(a \cdot x - \xi_{\rm b})^3}{24} + (a \cdot x - \xi_{\rm b}) & \text{for } a \cdot x \in (\xi_{\rm b}, \sqrt{2}), \end{cases} \tag{4.3a}$$

$$\nu_x \;=\; \begin{cases} \dfrac{1 - a \cdot \nabla u(x)}{2}\,\delta_{-a} + \dfrac{1 + a \cdot \nabla u(x)}{2}\,\delta_a & \text{for } a \cdot x \in (0, \xi_{\rm b}), \\ \delta_{\nabla u(x)} & \text{for } a \cdot x \in (\xi_{\rm b}, \sqrt{2}), \end{cases} \tag{4.3b}$$

provided we choose the boundary data $u_{\rm D} := u|_{\partial\Omega}$ with u just from (4.3a).

We use a regular triangulation $\mathcal{T}_d$ for the finite element mesh shown in figure 1.

We have implemented the convex mathematical program corresponding to the optimization problem $(\mathfrak{MP}_{d,\kappa})$ defined in (u, m) variables, where we represent the admissible function u by using a finite element linear interpolation basis, defined by the finite element mesh shown in 1. With the optimal vectors m^* so obtained, we first check either if they actually represent the algebraic moments of a two dimensional probability measure or not. If not we increase κ and try again. For this case we have stopped in $\kappa = 3$. Usually, for coercive polynomials, we obtain valid set of moments in a few steps, just taking κ a bit greater than the degree of φ_1.

Afterwards we have obtained a set of bi-dimensional moments, we use the marginal moments on the axis x and y to construct the probability measure that they represent. We must notice that this procedure is not possible in general cases, we

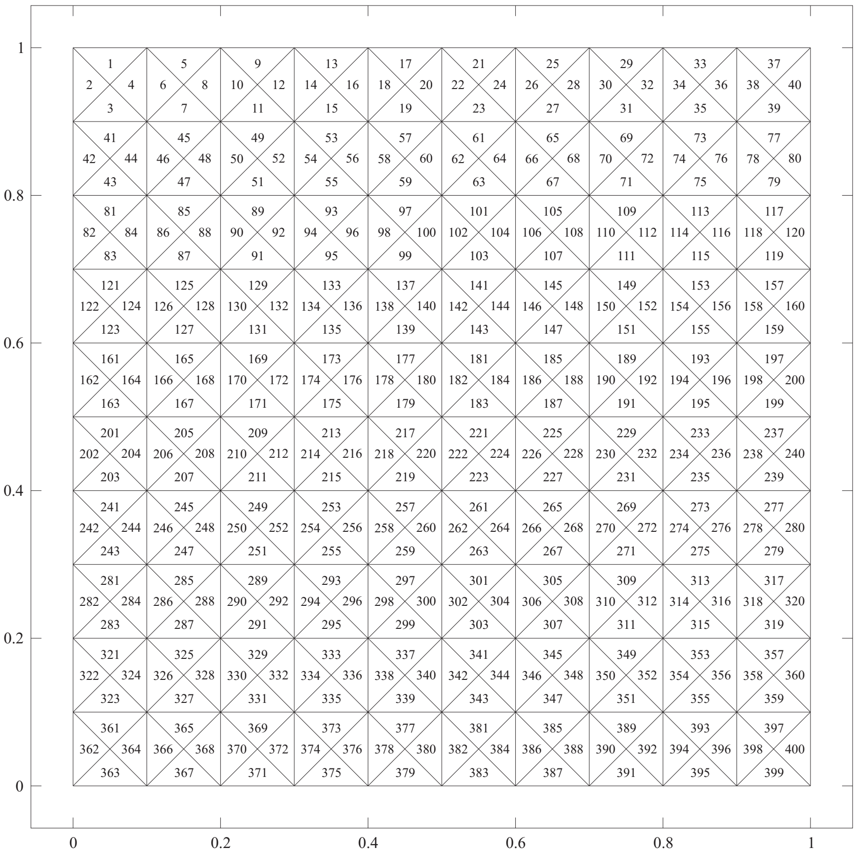

Figure 1: Finite element mesh

Fig5

success here because we obtain moments which represent the probability measures defining the convex hull of the non convex potential φ_1. This is an interesting application of Caratheosory's theorem. See [44, 45] for a clearer description of the algorithms that we use to calculate the optimal Young measure from the optimal moments obtained in $(\mathfrak{MP}_{d,\kappa})$.

Numerical results are shown in Table 1. Beside the number of every mesh-element, we show the optimal Young measure by specifying two probabilities and two bi-dimensional supports. Thus, in the heading of Table 1, *el* stands for element number, *pr* stands for probability and *sx1* stands for the x-coordinate of the supporting point corresponding to the probability *pr1*. Now the reader can easily grasp the meaning of all the items into the heading line of Table 1.

The optimal surface u^* shown in 2, has been calculated by using the corresponding coefficients of a first-order spline basis, defined over the the finite-element mesh shown in 1. Calculations have been done by implementing the model in Matlab code, where optimization routines have been linked to an Ampl interface, see [45] for a deeper description of all the minor details of this implementation.

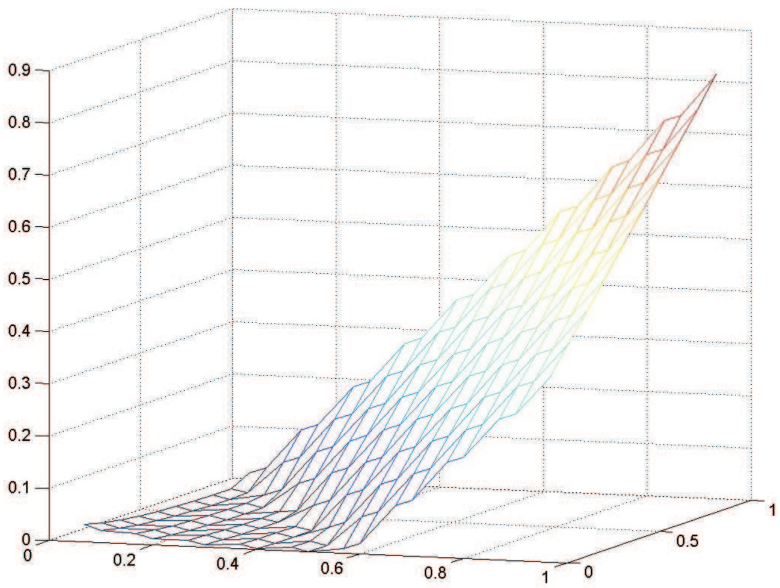

Figure 2: Optimal u^* based in the finite element mesh of 1. Fig4

5 Conclusions

In this paper we have shown the mathematical analysis of variational problems given in the general way $(\mathfrak{P})$ where the potential φ_1 can be expressed as a non-linear, non-convex multidimensional polynomial in the gradient of the admissible function u. We also propose a very specific numerical method based in convex optimization to find explicit generalized solutions defined in Young measures. This approach is very enlightening for everybody engaged with models in non-linear elasticity and other branches of mathematical physics implied with non-linear variational problems. A good deal of research follows now, to exploit numerically the methods and techniques proposed in this work.

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PrDis_1

Figure 3: Optimal bi-dimensional parametrized measures

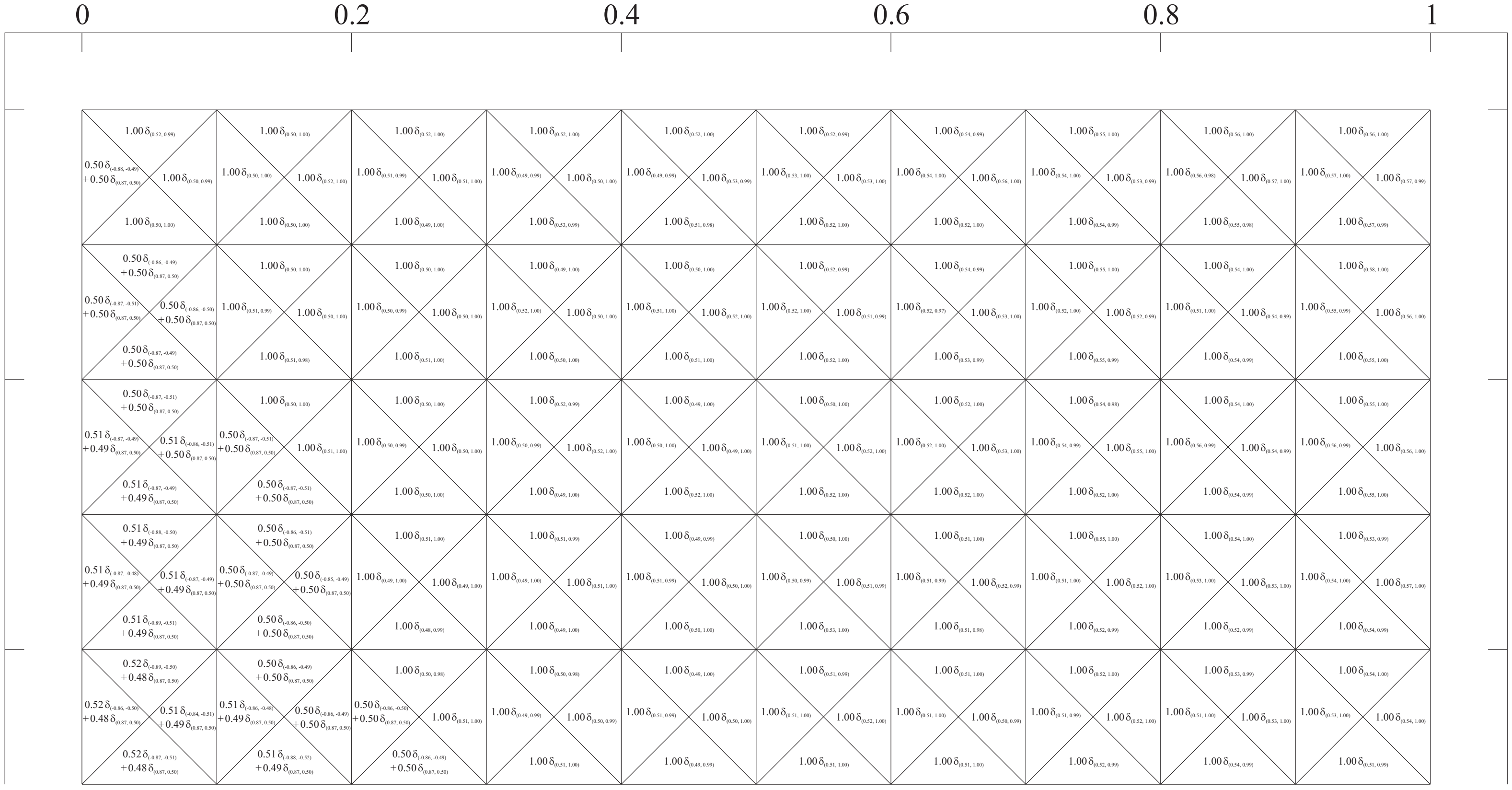
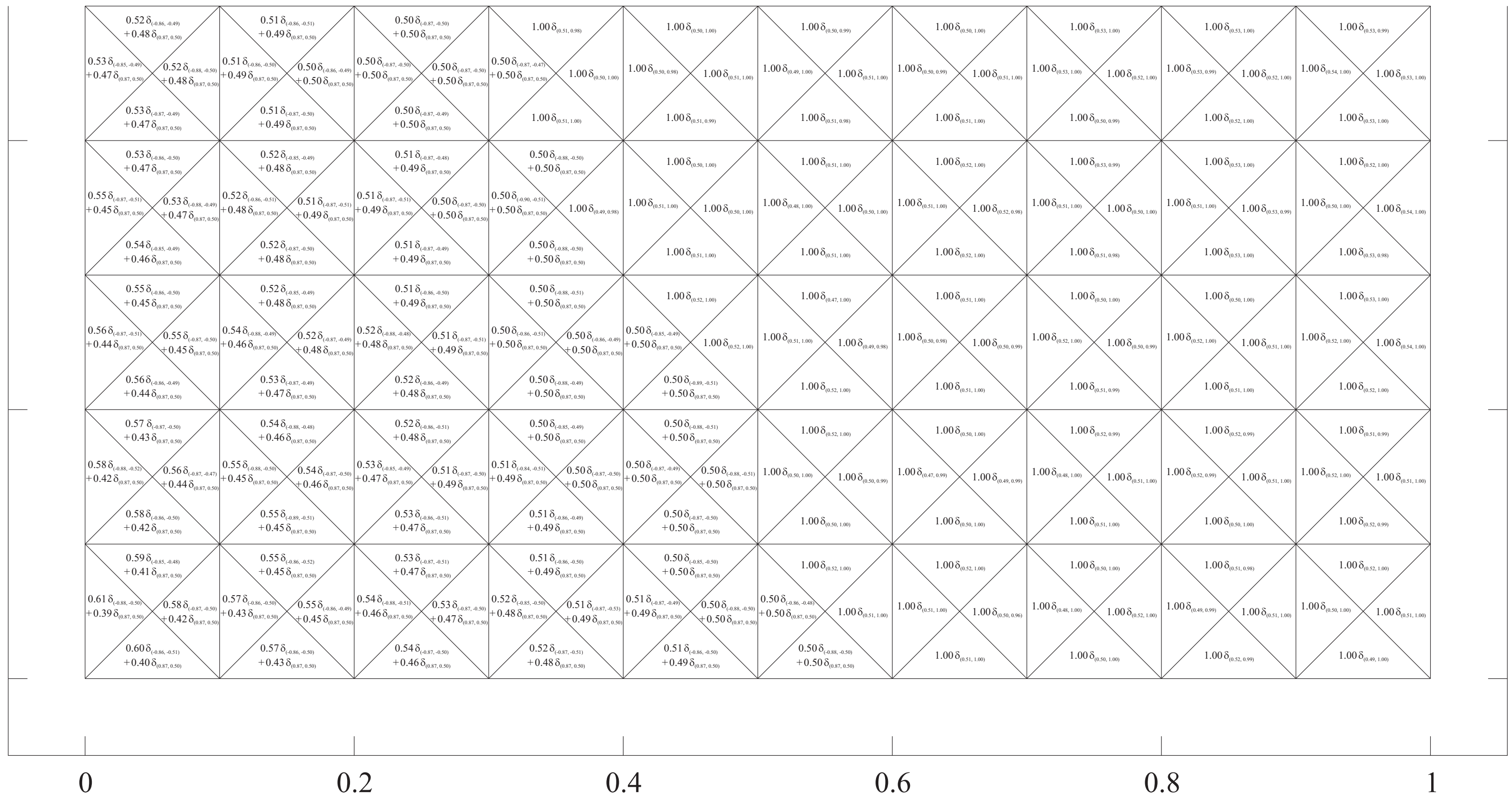


Figure 3: Optimal bi-dimensional parametrized measures (continued)



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