Analysis of Polyconvex Envelopes of Polynomial Expressions

R. Meziat
Departamento de Matemáticas
Universidad de los Andes
ICOTA-6
Ballarat, 2004

The author thanks: Research Grant 1204-05-13627, CT 282-2003, COLCIENCIAS

Introduction (1/2)

- Functionals of the form:

$$I(u) = \iint_{\Omega} \left\{ \phi\left(x, y; \vec{\nabla} u\right) + \psi\left(x, y; u\right) \right\} dx\, dy$$

- Admissible functions are displacement vector fields:

$$u : \Omega \to R^2$$

- Subject to particular contour conditions:

$$u = g \quad \text{in } \partial\Omega$$

Introduction (2/2)

- The gradient of u is a deformation matrix:

$$\vec{\nabla} u = \begin{pmatrix} \frac{\partial u_1}{\partial x} & \frac{\partial u_1}{\partial y} \\ \frac{\partial u_2}{\partial x} & \frac{\partial u_2}{\partial y} \end{pmatrix}$$

- Therefore, the energy potential is dependant on a 2x2 matrix:

$$\phi = \phi(A) \qquad A \in R^{2x2}$$

- ϕ is polyconvex if it depends on a convex way from matrix A and its determinant:

$$\phi(A) = h(A, \det A)$$

Polyconvexity (1/3)

- Polyconvexity was introduced by Ball in 1977*
- A function $f : R^5 \rightarrow R$ is polyconvex if there exists a convex function $h : R^5 \rightarrow \overline{R}$ such that

$$\phi(A) = h(T(A))$$

where $T : R^5 \rightarrow \overline{R}$ is dependant of the determinant of A and the components of A:

$$T(A) = T(A, \det(A))$$

* Dacorogna, B., *Direct methods in The Calculus of Variations*, Springer-Verlag, 1989.
Pedregal, P., *Variational Methods in Nonlinear Elasticity*, SIAM, 2000.

Polyconvexity (2/3)

- Some properties of polyconvexity:
  - If ϕ is convex, then it is polyconvex
  - When ϕ is a quadratic form it is polyconvex if and only if there exists a constant c such that:
  $$\phi(A) \geq c \det(A)$$
  - When g is a convex function where $g(0) = \min\{g(x) : x \geq 0\}$ then, the following is a polyconvex function:
  $$\phi(A) = g\left(\|A\|_2\right)$$
  - When $1 \leq \alpha \leq 3$ and g is a real convex function, the following is polyconvex:
  $$\phi(A) = \|A\|_2^{\alpha} + g(\det A)$$

Polyconvexity (3/3)

- If f is polyconvex in

$$\min_{u} \iint_{\Omega} f\left(x, y, u, \vec{\nabla} u\right) dx\, dy$$

  for every point (x,y) the existence of minimizers is assured
- Otherwise the polyconvex envolvent of f must be considered
- Is this approach to admit a formulation of semidefinite relaxations when f is described by polynomials?

Polyconvex Envolvents (1/6)

- The estimation of the polyconvex envolvent of a polynomial potential ϕ admits a formulation in terms of semidefinite programs
- The polyconvex envolvent is the bigger polyconvex functions that bounds ϕ from below
- Also, the polyconvex envolvent of ϕ is not to exceed its convex envolvent:

$$\phi_c \leq \phi_p$$

Polyconvex Envolvents (2/6)

- The estimation of the polyconvex envolvent of f for square matrixes of dimension 2 via convex combinations can be such that:

$$\phi_p(A) = \min \sum_{i=1}^{6} \lambda_i \phi\left(A^i\right)$$

- Where all convex combinations of six terms are considered that involucrate matrixes A^i of dimension 2x2 such that:

$$\sum_{i=1}^{6} \lambda_i A^i = A, \quad \sum_{i=1}^{6} \lambda_i \det\left(A^i\right) = \det(A)$$

Polyconvex Envolvents (3/6)

- Every convex combination can be replaced by a probability distribution in the 2x2 matrix space, such that the estimation of the polyconvex envolvent of function ϕ in the matrix A is

$$\phi_p(A) = \min \iint\iint_{Q \in R^4} \phi(Q) d\mu(Q)$$

$$\text{s.t.} \quad A = \iint\iint_{Q \in R^4} Q d\mu(Q),$$

$$\det(A) = \iint\iint_{Q \in R^4} \det(Q) d\mu(Q)$$

Polyconvex Envolvents (4/6)

- If function ϕ has polynomial structure

$$\phi(A) = \phi(x_1, x_2, x_3, x_4) = \sum_{0 \le i+j+k+l \le 2n} c_{ijkl} x_1^i x_2^j x_3^k x_4^l$$

$$\text{where} \quad A = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}.$$

Polyconvex Envolvents (5/6)

- The polyconvex envolvent estimation can be written as an optimization problem in measure as the mathematical program:

$$\phi_p(A) = \min_m \sum_{0 \le i+j+k+l \le 2n} c_{ijkl} m_{ijkl}$$

$$\text{s.t.} \quad A = \begin{pmatrix} m_{1000} & m_{0100} \\ m_{0010} & m_{0001} \end{pmatrix},$$

$$\det(A) = m_{1000} m_{0001} - m_{0010} m_{0100}$$

Polyconvex Envolvents (6/6)

- In which the design variables m_{ijkl} must represent the algebraic moments of a probability measure in R^4. This implies an additional restriction in the form of a matrix inequality

$$m_{ijkl} = \int x_1^i x_2^j x_3^k x_4^l \, d\mu\left(x_1^i, x_2^j, x_3^k, x_4^l\right)$$

- The restriction matrixes must be semidefinite positive

Restriction Matrixes (1/2)

- Given the functions following functions

$$x_1^i x_2^j x_3^k x_4^l \qquad \text{where} \qquad 0 \le i + j + k + l \le n$$

  they are written as a list in the first row and column of a table
- To fill every position in the matrix multiply every element of the first row and the first column between them: $x_1^{i+i'} x_2^{j+j'} x_3^{k+k'} x_4^{l+l'} \qquad \text{where} \qquad 0 \le i + j + k + l \le n$

$$0 \le i' + j' + k' + l' \le n$$

	1	x_1	$\dots$	x_4	x_1^2	$\dots$	x_4^2	x_1x_2	$\dots$	x_3x_4
1	1	x_1	$\dots$	x_4	x_1^2	$\dots$	x_4^2	x_1x_2	$\dots$	x_3x_4
x_1	x_1	x_1^2	$\dots$	x_1x_4	x_1^3	$\dots$	$x_1x_4^2$	$x_1^2x_2$	$\dots$	$x_1x_3x_4$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_4	x_4	x_1x_4	$\dots$	x_4^2	$x_1^2x_4$	$\dots$	x_4^3	$x_1x_2x_4$	$\dots$	$x_3x_4^2$
x_1^2	x_1^2	x_1^3	$\dots$	$x_1^2x_4$	x_1^4	$\dots$	$x_1^2x_4^2$	$x_1^3x_2$	$\dots$	$x_1^2x_3x_4$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_4^2	x_4^2	$x_1x_4^2$	$\dots$	x_4^3	$x_1^2x_4^2$	$\dots$	x_4^4	$x_4^2x_1x_2$	$\dots$	$x_4^3x_3$
x_1x_2	x_1x_2	$x_1^2x_2$	$\dots$	$x_1x_2x_4$	$x_1^3x_2$	$\dots$	$x_4^2x_1x_2$	$x_1^2x_2^2$	$\dots$	$x_1x_2x_3x_4$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_3x_4	x_3x_4	$x_1x_3x_4$	$\dots$	$x_3x_4^2$	$x_1^2x_3x_4$	$\dots$	$x_4^3x_3$	$x_1x_2x_3x_4$	$\dots$	$x_3^2x_4^2$

Restriction Matrixes (2/2)

- Change the functions for their respective moment

$$x_1^i x_2^j x_3^k x_4^l \rightarrow m_{ijkl}$$

- The resulting matrix must be semidefinite positive

$$M = \begin{bmatrix} m_{0000} & m_{1000} & \cdots & m_{0001} & m_{2000} & \cdots & m_{0002} & m_{1100} & \cdots & m_{0011} \\ m_{1000} & m_{2000} & \cdots & m_{1001} & m_{3000} & \cdots & m_{1002} & m_{2100} & \cdots & m_{1011} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ m_{0001} & m_{1001} & \cdots & m_{0002} & m_{2001} & \cdots & m_{0003} & m_{1101} & \cdots & m_{0012} \\ m_{2000} & m_{3000} & \cdots & m_{2001} & m_{4000} & \cdots & m_{2002} & m_{31000} & \cdots & m_{2011} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ m_{0002} & m_{2100} & \cdots & m_{1101} & m_{3100} & \cdots & m_{1102} & m_{22000} & \cdots & m_{1111} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ m_{0011} & m_{1011} & \cdots & m_{0012} & m_{2011} & \cdots & m_{0013} & m_{1111} & \cdots & m_{0022} \end{bmatrix}$$

Polyconvex Env. Conclusion

- To find the polyconvex envolvent of a function solve the semidefinite program in the standar form:

$$\text{Minimize} \quad \sum_{0 \le i+j+k+l \le 4} c_{ijkl} m_{ijkl}$$

$$\text{s.t.} \quad R + M_2 \ge 0,$$

$$a_1 a_4 - a_2 a_3 = m_{1001} - m_{0110},$$

$$m_{0000} = 1$$

$$M_2 = \begin{bmatrix} 0 & 0 & \dots & 0 & m_{2000} & \dots & m_{0002} & m_{1100} & \dots & m_{0011} \\ 0 & m_{2000} & \dots & m_{1001} & m_{3000} & \dots & m_{1002} & m_{2100} & \dots & m_{1011} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & m_{1001} & \dots & m_{0002} & m_{2001} & \dots & m_{0003} & m_{1101} & \dots & m_{0012} \\ m_{2000} & m_{3000} & \dots & m_{2001} & m_{4000} & \dots & m_{2002} & m_{31000} & \dots & m_{2011} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ m_{0002} & m_{2100} & \dots & m_{1101} & m_{3100} & \dots & m_{1102} & m_{22000} & \dots & m_{1111} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ m_{0011} & m_{1011} & \dots & m_{0012} & m_{2011} & \dots & m_{0013} & m_{1111} & \dots & m_{0022} \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & a_{11} & \dots & a_{22} & 0 & \dots & 0 & 0 & \dots & 0 \\ a_{11} & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{22} & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{bmatrix}$$

Example 1 (1/2) $\phi(A) = \left(1 - \det(A)^2\right)^2$

- As instance take

$$\phi(A) = \left(1 - \det(A)^2\right)^2$$

$$A = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$$

- An 8th degree polynomial, this implies a 70x70 restriction matrix.

Example 1 (2/2) $\phi(A) = \left(1 - \det(A)^2\right)^2$

- The polyconvex envolvment is calculated at

$$A = \begin{bmatrix} 1 & 0.1 \\ 0.5 & 0.3 \end{bmatrix}$$

- The value of the polyconvex envolvent at this point is $\phi_c(A)$=0.1 ($\phi(A)$=0.8789)

Moment	Value	Moment	Value	Moment	Value	Moment	Value	Moment	Value
m_{2000}	1.6119	m_{1100}	0.5581	m_{0310}	0.0782	m_{2201}	0.2453	m_{2310}	0.0831
m_{1010}	0.1116	m_{2100}	0.7376	m_{0301}	0.1589	m_{1211}	0.0283	m_{2301}	0.1899
m_{2010}	0.1430	m_{3100}	0.9064	m_{0400}	1.8542	m_{1202}	0.3127	m_{2400}	2.4029
m_{3010}	0.1750	m_{0110}	0.0732	m_{4010}	0.2301	m_{2300}	0.7326	m_{1050}	0.1001
m_{0020}	0.7531	m_{1110}	0.0842	m_{3020}	0.7846	m_{1310}	0.0828	m_{1041}	0.0857
m_{1020}	0.4889	m_{2110}	0.0944	m_{2030}	0.1230	m_{1301}	0.1636	m_{1032}	0.0413
m_{2020}	1.0382	m_{0120}	0.1910	m_{1040}	0.5653	m_{1400}	0.8929	m_{1023}	0.0781
m_{0030}	0.1027	m_{1120}	0.1789	m_{5000}	3.3920	m_{0050}	0.1309	m_{1014}	0.0611
m_{1030}	0.0980	m_{0130}	0.0762	m_{4001}	0.6785	m_{0041}	0.1366	m_{1140}	0.1396
m_{0040}	1.5515	m_{0101}	0.1741	m_{3011}	0.1042	m_{0032}	0.0511	m_{1131}	0.4134
m_{4000}	3.3459	m_{1101}	0.1975	m_{2021}	0.1260	m_{0023}	0.1304	m_{1122}	0.0428
m_{1001}	0.3232	m_{2101}	0.2427	m_{1031}	0.0649	m_{0014}	0.0659	m_{1113}	0.4358
m_{2001}	0.4216	m_{0111}	0.0775	m_{3002}	0.8144	m_{0140}	0.1791	m_{1230}	0.0717
m_{3001}	0.5161	m_{1111}	0.2035	m_{2012}	0.0859	m_{0131}	0.0947	m_{1221}	0.0382
m_{0011}	0.0564	m_{0121}	0.0407	m_{1022}	0.2142	m_{0122}	0.0772	m_{1212}	0.0216
m_{1011}	0.0716	m_{0102}	0.2473	m_{2003}	0.3183	m_{0113}	0.1086	m_{1320}	0.1896
m_{2011}	0.0752	m_{1102}	0.2292	m_{1013}	0.0896	m_{0230}	0.0796	m_{1311}	0.3985
m_{0021}	0.1210	m_{0112}	0.0446	m_{1004}	0.4346	m_{0221}	0.0640	m_{1410}	0.0727
m_{1021}	0.0970	m_{0103}	0.1294	m_{4100}	1.2210	m_{0212}	0.0303	m_{5020}	1.2329
m_{0031}	0.0491	m_{0200}	0.9755	m_{3110}	0.1209	m_{0320}	0.2319	m_{4030}	0.1736
m_{0002}	0.7887	m_{1200}	0.7487	m_{2120}	0.2313	m_{0311}	0.0869	m_{3040}	0.7066
m_{1002}	0.4960	m_{2200}	1.3631	m_{1130}	0.0827	m_{0410}	0.0820	m_{2050}	0.1240
m_{2002}	0.9152	m_{0210}	0.0689	m_{3101}	0.2989	m_{0104}	0.2371	m_{6010}	0.3678
m_{0012}	0.0645	m_{1210}	0.0677	m_{2111}	0.0716	m_{0203}	0.1803	m_{5011}	0.1606
m_{1012}	0.0673	m_{0220}	0.6000	m_{1121}	0.0359	m_{0302}	0.2641	m_{4021}	0.1797
m_{0022}	0.7093	m_{0201}	0.2068	m_{2102}	0.3039	m_{0401}	0.2285	m_{3031}	0.0787
m_{0003}	0.2850	m_{1201}	0.1961	m_{1112}	0.0484	m_{0500}	0.8283	m_{2041}	0.1058
m_{1003}	0.2407	m_{0211}	0.0304	m_{1103}	0.1329	m_{0005}	0.3691	m_{4012}	0.1363
m_{0013}	0.0659	m_{0202}	0.8004	m_{3200}	1.1868	m_{5010}	0.2801	m_{3022}	0.2813
m_{0004}	1.5301	m_{0300}	0.5851	m_{2210}	0.0787	m_{4020}	2.3482	m_{2032}	0.0519
m_{3000}	1.8111	m_{1300}	0.5640	m_{1220}	0.3025	m_{3030}	0.1349	m_{3013}	0.1354

Example 2 (1/2) $\phi(A) = \left(1 + \det(A)^2\right)^2$

- As instance take

$$\phi(A) = \left(1 + \det(A)^2\right)^2$$

$$A = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$$

- An 8th degree polynomial, this implies a 70x70 restriction matrix.
- This polynomial is polyconvex by definition

Example 2 (2/2) $\phi(A) = \left(1 + \det(A)^2\right)^2$

- The polyconvex envolvment is calculated at

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$$

- The value of the polyconvex envolvent at this point is $\phi_c(A)=1$ ($\phi(A)=1$)

Moment	Value	Moment	Value
m_{0200}	$3{,}383e+003$	m_{1210}	$1{,}097e+005$
m_{1100}	$7{,}927e+000$	m_{0310}	$1{,}717e+005$
m_{1001}	$-4{,}499e-008$	m_{0211}	$-9{,}163e+004$
m_{0101}	$1{,}428e+001$	m_{0220}	$6{,}307e+006$
m_{0002}	$3{,}426e+003$	m_{2101}	$1{,}097e+005$
m_{2000}	$3{,}372e+003$	m_{1102}	$-9{,}164e+004$
m_{1010}	$5{,}687e+000$	m_{3100}	$5{,}802e+005$
m_{0110}	$9{,}999e-001$	m_{2110}	$1{,}390e+005$
m_{0011}	$1{,}296e+001$	m_{1111}	$6{,}307e+006$
m_{0020}	$3{,}394e+003$	m_{1120}	$3{,}536e+005$
m_{1200}	$1{,}823e+003$	m_{2002}	$6{,}307e+006$
m_{2100}	$3{,}236e+003$	m_{1003}	$7{,}303e+004$
m_{2001}	$4{,}223e+003$	m_{3001}	$1{,}356e+005$
m_{1101}	$2{,}212e+003$	m_{2011}	$3{,}536e+005$
m_{1002}	$7{,}030e+002$	m_{1012}	$5{,}417e+004$
m_{3000}	$3{,}356e+003$	m_{1021}	$1{,}301e+005$
m_{2010}	$4{,}724e+003$	m_{0103}	$2{,}099e+005$
m_{1110}	$4{,}224e+003$	m_{0112}	$7{,}646e+004$
m_{1011}	$1{,}488e+003$	m_{0121}	$5{,}419e+004$
m_{1020}	$1{,}681e+003$	m_{0004}	$3{,}213e+007$
m_{0300}	$7{,}025e+003$	m_{0013}	$1{,}560e+005$
m_{0201}	$9{,}443e+003$	m_{0022}	$1{,}367e+007$
m_{0102}	$3{,}354e+003$	m_{4000}	$3{,}084e+007$
m_{0210}	$2{,}214e+003$	m_{3010}	$2{,}163e+005$
m_{0111}	$7{,}090e+002$	m_{2020}	$1{,}434e+007$
m_{0120}	$1{,}491e+003$	m_{1030}	$-4{,}337e+004$
m_{0012}	$5{,}155e+003$	m_{0130}	$1{,}335e+005$
m_{0021}	$9{,}339e+003$	m_{0031}	$2{,}077e+005$
m_{0030}	$1{,}049e+004$	m_{0040}	$3{,}153e+007$
m_{0003}	$2{,}124e+004$	m_{0301}	$-5{,}593e+004$
m_{0400}	$3{,}126e+007$	m_{0202}	$1{,}427e+007$
m_{1300}	$-2{,}204e+005$	m_{2200}	$1{,}412e+007$
m_{1201}	$1{,}684e+005$		