Analysis of Polyconvex Envelopes of Polynomial Expressions

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Introduction (1/2)

- Functionals of the form:
  \[ I(u) = \iint_{\Omega} \{ \phi(x, y; \nabla u) + \psi(x, y; u) \} dx \, dy \]

- Admissible functions are displacement vector fields:
  \[ u : \Omega \to \mathbb{R}^2 \]

- Subject to particular contour conditions:
  \[ u = g \quad \text{in} \quad \partial \Omega \]
Introduction (2/2)

- The gradient of $u$ is a deformation matrix:

$$\nabla u = \begin{pmatrix}
\frac{\partial u_1}{\partial x} & \frac{\partial u_1}{\partial y} \\
\frac{\partial u_2}{\partial x} & \frac{\partial u_2}{\partial y}
\end{pmatrix}$$

- Therefore, the energy potential is dependant on a 2x2 matrix:

$$\phi = \phi(A) \quad A \in R^{2 \times 2}$$

- $\phi$ is polyconvex if it depends on a convex way from matrix $A$ and its determinant:

$$\phi(A) = h(A, \det A)$$
Polyconvexity (1/3)

- Polyconvexity was introduced by Ball in 1977*
- A function $f : \mathbb{R}^5 \to \mathbb{R}$ is polyconvex if there exists a convex function $h : \mathbb{R}^5 \to \mathbb{R}$ such that
  
  $$\phi(A) = h(T(A))$$

  where $T : \mathbb{R}^5 \to \mathbb{R}$ is dependant of the determinant of $A$ and the components of $A$:

  $$T(A) = T(A, \det(A))$$

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Polyconvexity (2/3)

Some properties of polyconvexity:

- If $\phi$ is convex, then it is polyconvex.
- When $\phi$ is a quadratic form it is polyconvex if and only if there exists a constant $c$ such that:
  $$\phi(A) \geq c \det(A)$$
- When $g$ is a convex function where $g(0) = \min\{g(x) : x \geq 0\}$ then, the following is a polyconvex function:
  $$\phi(A) = g(\|A\|_2)$$
- When $1 \leq \alpha \leq 3$ and $g$ is a real convex function, the following is polyconvex:
  $$\phi(A) = \|A\|_2^\alpha + g(\det A)$$
Polyconvexity (3/3)

- If $f$ is polyconvex in
  \[
  \min_u \iint_{\Omega} f(x, y, u, \nabla u) \, dx \, dy
  \]
  for every point $(x, y)$ the existence of minimizers is assured

- Otherwise the polyconvex envolvent of $f$ must be considered

- Is this approach to admit a formulation of semidefinite relaxations when $f$ is described by polynomials?
Polyconvex Envolvents (1/6)

- The estimation of the polyconvex envolvent of a polynomial potential $\phi$ admits a formulation in terms of semidefinite programs.
- The polyconvex envolvent is the bigger polyconvex functions that bounds $\phi$ from below.
- Also, the polyconvex envolvent of $\phi$ is not to exceed its convex envolvent:

$$\phi_c \leq \phi_p$$
Polyconvex Envolvents (2/6)

- The estimation of the polyconvex envolvent of \( f \) for square matrixes of dimension 2 via convex combinations can be such that:
  \[
  \phi_p(A) = \min \sum_{i=1}^{6} \lambda_i \phi(A^i)
  \]

- Where all convex combinations of six terms are considered that involucrate matrixes \( A^i \) of dimension 2x2 such that:
  \[
  \sum_{i=1}^{6} \lambda_i A^i = A, \quad \sum_{i=1}^{6} \lambda_i \det(A^i) = \det(A)
  \]
Polyconvex Envolvents (3/6)

Every convex combination can be replaced by a probability distribution in the 2x2 matrix space, such that the estimation of the polyconvex envolvent of function $\phi$ in the matrix $A$ is

$$
\phi_p(A) = \min \int \int \int \int_{Q \in R^4} \phi(Q)d\mu(Q)
$$

s.t. $A = \int \int \int \int_{Q \in R^4} Q d\mu(Q)$,

$$
det(A) = \int \int \int \int_{Q \in R^4} det(Q)d\mu(Q)
$$
Polyconvex Envolvents (4/6)

If function $\phi$ has polynomial structure

$$\phi(A) = \phi(x_1, x_2, x_3, x_4) = \sum_{0 \leq i+j+k+l \leq 2n} c_{ijkl} x_1^i x_2^j x_3^k x_4^l$$

where $A = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$. 
Polyconvex Envolvents (5/6)

The polyconvex envolvent estimation can be written as an optimization problem in measure as the mathematical program:

\[
\phi_p(A) = \min_m \sum_{0 \leq i+j+k+l \leq 2n} c_{ijkl} m_{ijkl}
\]

s.t. \( A = \begin{pmatrix} m_{1000} & m_{0100} \\ m_{0010} & m_{0001} \end{pmatrix} \),

\[
\det(A) = m_{1000} m_{0001} - m_{0010} m_{0100}
\]
Polyconvex Envolvents (6/6)

- In which the design variables $m_{ijkl}$ must represent the algebraic moments of a probability measure in $\mathbb{R}^4$. This implies an additional restriction in the form of a matrix inequality

$$m_{ijkl} = \int x_1^i x_2^j x_3^k x_4^l \, d\mu(x_1^i, x_2^j, x_3^k, x_4^l)$$

- The restriction matrixes must be semidefinite positive
Restriction Matrixes (1/2)

- Given the functions following functions
  \[ x_1^i x_2^j x_3^k x_4^l \]
  they are written as a list in the first row and column of a table.

- To fill every position in the matrix multiply every element of the first row and
  the first column between them:
  \[ x_1^{i+j'} x_2^{j+k'} x_3^{k+l'} \]
  where \[ 0 \leq i + j + k + l \leq n \]
  \[ 0 \leq i' + j' + k' + l' \leq n \]

\[
\begin{array}{cccccccc}
1 & x_1 & \ldots & x_4 & x_1^2 & \ldots & x_4^2 & \ldots & x_3x_4 \\
x_1 & x_1 & x_4 & x_1^2 & \ldots & x_4^2 & \ldots & x_3x_4 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
x_4 & x_1x_4 & x_4^2 & x_1x_4 & x_4^2 & \ldots & x_3x_4 \\
x_1^2 & x_1^2 & x_1^2 & x_1^2 & x_1^2 & \ldots & x_1x_3x_4 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
x_4^2 & x_4^2 & x_4^2 & x_4^2 & x_4^2 & \ldots & x_4^2 \\
x_1x_2 & x_1x_2 & x_1x_2 & x_1x_2 & x_1x_2 & \ldots & x_1x_3x_4 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
x_3x_4 & x_3x_4 & x_3x_4 & x_3x_4 & x_3x_4 & \ldots & x_3x_4 \\
\end{array}
\]
Restriction Matrixes (2/2)

- Change the functions for their respective moment

\[ x_1^i x_2^j x_3^k x_4^l \rightarrow m_{ijkl} \]

- The resulting matrix must be semidefinite positive

\[ M = \begin{bmatrix}
  m_{0000} & m_{1000} & \cdots & m_{0001} & m_{2000} & \cdots & m_{0002} & m_{1100} & \cdots & m_{0011} \\
  m_{1000} & m_{2000} & \cdots & m_{1001} & m_{3000} & \cdots & m_{1002} & m_{2100} & \cdots & m_{1011} \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
  m_{0001} & m_{1001} & \cdots & m_{0002} & m_{2001} & \cdots & m_{0003} & m_{1101} & \cdots & m_{0012} \\
  m_{2000} & m_{3000} & \cdots & m_{2001} & m_{4000} & \cdots & m_{2002} & m_{3100} & \cdots & m_{2011} \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
  m_{0002} & m_{2100} & \cdots & m_{1101} & m_{3100} & \cdots & m_{1102} & m_{2200} & \cdots & m_{1111} \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
  m_{0011} & m_{1011} & \cdots & m_{0012} & m_{2011} & \cdots & m_{0013} & m_{1111} & \cdots & m_{0022}
\end{bmatrix} \]
Polyconvex Env. Conclusion

- To find the polyconvex envelopent of a function solve the semidefinite program in the standard form:

\[
\text{Minimize } \sum_{0 \leq i+j+k+l \leq 4} c_{ijkl} m_{ijkl}
\]

s.t. \( R + M_2 \geq 0, \)
\( a_1 a_4 - a_2 a_3 = m_{1001} - m_{0110}, \)
\( m_{0000} = 1 \)
Example 1 (1/2) \( \phi(A) = \left(1 - \det(A)^2\right)^2 \)

- As instance take

\[
\phi(A) = \left(1 - \det(A)^2\right)^2
\]

\[
A = \begin{bmatrix}
    x_1 & x_2 \\
    x_3 & x_4
\end{bmatrix}
\]

- An 8th degree polynomial, this implies a 70x70 restriction matrix.
Example 1 (2/2) \( \phi(A) = \left(1 - \det(A)^2\right)^2 \)

- The polyconvex envelopment is calculated at

\[
A = \begin{bmatrix}
1 & 0.1 \\
0.5 & 0.3 \\
\end{bmatrix}
\]

- The value of the polyconvex envelopent at this point is \( \phi_c(A) = 0.1 \) \( (\phi(A) = 0.8789) \)

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Example 2 (1/2) \[ \phi(A) = \left(1 + \det(A)^2\right)^2 \]

- As instance take

\[ \phi(A) = \left(1 + \det(A)^2\right)^2 \]

\[ A = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \]

- An 8th degree polynomial, this implies a 70x70 restriction matrix.

- This polynomial is polyconvex by definition
Example 2 (2/2) \( \phi(A) = \left(1 + \det(A)^2\right)^2 \)

- The polyconvex envolvment is calculated at
  \[
  A = \begin{bmatrix}
  1 & 2 \\
  3 & 6
  \end{bmatrix}
  \]

- The value of the polyconvex envolvent at this point is \( \phi_c(A) = 1 \) \( (\phi(A) = 1) \)