# The Method of Moments

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# General Mathematical Program

$$\min_{t \in \Omega} f(t)$$

$f$ continuous and bounded from below

$\Omega \subset R^n$

# Convex Envelopes

$$\min f_c = global \min f$$

# Caratheodory's Theorem

$$f_c(t) = \min \sum_{k=1}^{n+1} \lambda_k f(t_k)$$

$$s.a$$

$$t = \sum_{k=1}^{n+1} \lambda_k t_k$$

$$1 = \sum_{k=1}^{n+1} \lambda_k$$

$$\lambda_k \geq 0 \ \forall \ k = 1, \ldots, n+1$$

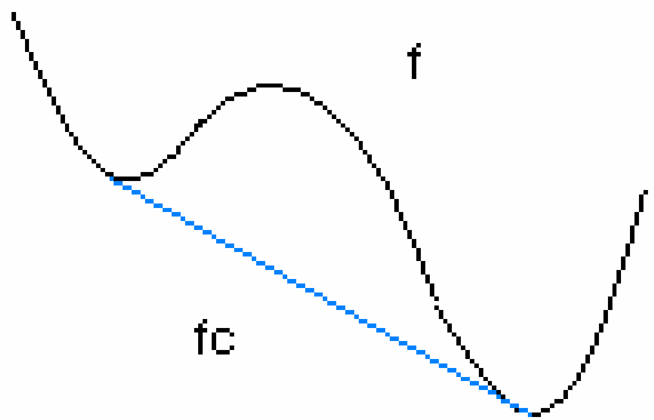
# Convex Analysis

$$Epi\left(f_c\right) = \overline{co\left(Epi\left(f\right)\right)}$$

$co$ : convex hull

$bar$ : closure

# Graphical Illustration




- Function
- Convex Envelope

# Probability

- Caratheodory's Theorem
- Convex Combinations as Discrete Probability Distributions

$$\sum_{k=1}^{n+1} \lambda_k f(t_k) = \sum_{k=1}^{m} \lambda_k f(t_k)$$

$$\sum_{k=1}^{m} \lambda_k f(t_k) = \int_{\Omega} f(s) dv(s)$$

# Measure

- Discrete Distributions approximate any Regular Borel Measures

$$\sum_{k=1}^{m} \lambda_k f(t_k) \to \int_{\Omega} f(s)\,dv(s)$$

# Convex Envelopes
# Alternative Estimation with Measures

$$f_c(t) = \min \int_{\Omega} f(s) dv(s)$$

$s.t.$

$$t = \int_{\Omega} s \, dv(s)$$

$v \in P(\Omega)$ : set of probability measures in $\Omega$

# Global Optimization

$$global\ \min_{t\in\Omega} f(t) = \min_{t\in\Omega} f_c(t)$$

$$global\ \min_{t\in\Omega} f(t) = \min_{v\in P(\Omega)} \int_{\Omega} f(s)\,dv(s)$$

# Equivalent Problem

$$\min_{v \in P(\Omega)} \langle f, v \rangle$$

where

$$\langle f, v \rangle = \int_{\Omega} f(s)\, dv(s)$$

is a linear objective function

defined in a convex feasible set $P(\Omega)$

# Theorem 1

$$\left\langle f, v^* \right\rangle = \min_{v \in P(\Omega)} \left\langle f, v \right\rangle$$

if and only if

$v^* \in P(G)$

$G$ : set of global minima of $f$ in $\Omega$

$P(G)$: set of probability measures supported in $G$

# Decomposition in Moments

$$f(t) = \sum_{i=1}^{k} c_i \psi_i(t)$$

$\psi_1, \ldots, \psi_k$ : basis of continuous functions in $\Omega$

$$\langle f, v \rangle = \sum_{i=1}^{k} c_i m_i$$

$(m_1, \ldots, m_k)$: vector of generalized moments in $R^k$

# New Convex Program Equivalent

- Linear objective function
- Convex Feasible Set

$$\langle f, v \rangle = \sum_{i=1}^{k} c_i m_i = c \cdot m \qquad : \text{dot product in } R^k$$

$$v \to \langle (\psi_1, \ldots, \psi_k), v \rangle \qquad : P(\Omega) \to R^k \qquad \text{linear}$$

# New Convex Program Equivalent

$$\min_{m \in \overline{V}} c \cdot m$$

$$\overline{V} \subset R^k$$

$$\overline{V} : convex\ feasible\ set$$

# Theorem 2

- Every solution of the equivalent program (1) is linked with a global minima of the non convex global optimization program (2)

$$(1) \quad \min_{\overline{V}} c \cdot m$$

$$(2) \quad \min_{\Omega} f(t)$$

# Theorem 3

$t^* \in G$

when

$\left(\psi_1\left(t^*\right), \ldots, \psi_k\left(t^*\right)\right) = \left(m_1^*, \ldots, m_k^*\right)$

and $m^*$ is an extreme point of the solution set of

$\min_{\overline{V}} c \cdot m$

# Theorem 4

- Every solution $m^*$ of the convex program (1) is characterized by the following equality

$$m_i^* = \lambda_1 \psi_i\left(t_1^*\right) + \cdots + \lambda_{k+1} \psi_i\left(t_{k+1}^*\right)$$

$$i = 1, \cdots, k$$

$$1 = \lambda_1 + \cdots + \lambda_{k+1}$$

$$\lambda_i \geq 0$$

$$t^*_i \in G$$

# Convex Combination

- Notice the convex combination form in the previous equation

$$m^* = \lambda_1 \Psi\left(t^*{}_1\right) + \cdots + \lambda_{k+1} \Psi\left(t^*{}_{k+1}\right)$$

$$where$$

$$\Psi = \left(\psi_1, \ldots, \psi_k\right)$$

# Moments Form

- Notice the moments form of the previous equation

$$m^*_i = \int_{\Omega} \psi_i(s)\,dv(s) \quad i = 1, \dots, k$$

$$v = \sum_{i=1}^{k+1} \lambda_i \delta_{t_i^*}$$

# Theorem 5

- Every solution *m** of the Program (1) can be solved as a Problem of Moments by a discrete measure supported in G.

$$m^* = \int_\Omega \Psi(s)\,dv(s) \quad i = 1,\ldots,k$$

$$v \text{ discrete in } P(G)$$

# Classical Moment Problems

- Hamburger
- Stieltjes
- Hausdorff
- Trigonometric

$$1, t, \ldots, t^k$$

$$R, [0, \infty), [a, b]$$

$$e^{-jkt}, \ldots, e^{jkt}$$

$$[0, 2\pi]$$

# General Solution

$$M = \left\{ \int_{\Omega} \Psi(s)\,dv(s) : v \text{ medida positiva en } \Omega \right\}$$

$$cono\ momentos$$

$$P = \left\{ c \in R^k : \sum_{i=1}^{k} c_i \psi_i(t) \geq 0\ en\ \Omega \right\}$$

$$cono\ de\ funciones\ positivas$$

$$\overline{M} = P^*$$

# Particular Solutions

- Hamburger

$$\text{Positive Semidefinite Hankel Form}$$

$$k = 2n$$

$$H = \left(m_{i+j}\right)_{i,j=0,\dots,n}$$

- Trigonometric

$$\text{Positive Semidefinite Toeplitz Form}$$

$$k = n$$

$$H = \left(m_{i-j}\right)_{i,j=0,\dots,n}$$

# Classical Characterizations of Positive Polynomials

- Algebraic

$$q(t) \geq 0 \; in \; R, \; if \;\; and \; only \; if$$

$$q(t) = \left( \sum_{i=0}^{n} a_i t^i \right)^2 + \left( \sum_{i=0}^{n} b_i t^i \right)^2$$

- Trigonometric

$$q(t) \geq 0 \; in \; [0, 2\pi], \; if \;\; and \; only \; if$$

$$q(t) = \left| \sum_{j=0}^{n} a_j e^{ijt} \right|^2$$

# Quadratic forms

- Algebraic

$$q(t)=\sum_{i=0}^{n}\sum_{j=0}^{n}a_i a_j t^{i+j}+\sum_{i=0}^{n}\sum_{j=0}^{n}b_i b_j t^{i+j}$$

- Trigonometric

$$q(t)=\sum_{j=0}^{n}\sum_{j'=0}^{n}a_j \overline{a_{j'}} e^{i(j-j')t}$$

# Semidefinite Relaxations

- Algebraic

$$\min_R \sum_{i=0}^{2n} c_i t^i \quad \text{equivalent to}$$

$$\min \sum_{i=0}^{2n} c_i m_i \;\; s.t. \;\; \left(m_{i+j}\right)_{i,j=0,\ldots,n} \geq 0 \; and \; m_0 = 1$$

# Semidefinite Relaxations

- Trigonometric

$$\min_{S^1} \sum_{i=-n}^{n} c_i z^i \text{ equivalent to}$$

$$\min \sum_{i=-n}^{n} c_i m_i \ s.t. \ \left(m_{i-j}\right)_{i,j=0,\ldots,n} \geq 0 \ and \ m_0 = 1$$

# Recovering Global Minima

Seminidefinite Program

$$\min \sum_{i=0}^{2n} c_i m_i$$

$$\text{s.t.} \left(m_{i+j}\right)_{i,j=0,\dots,n} \geq 0 \; and \; m_0 = 1$$

$$solution \; m^*$$

$$roots \; of \;\; p(t) = \begin{vmatrix} m_0^* & m_1^* & \cdots & m_j^* \\ & & \cdots & \\ m^*_{j-1} & m_j^* & \cdots & m^*_{2j-1} \\ 1 & t & \cdots & t^j \end{vmatrix}$$

are global minima of $q(t) = \sum_{i=0}^{2n} c_i t^i$